ANNALES D'ÉCONOMIE ET DE STATISTIQUE. – N° 85 – 2007

Diffusion Processes with Polynomial Eigenfunctions

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ABSTRACT. – The aim of this paper is to characterize the one-dimensional stochastic differential equations, for which the eigenfunctions of the infinitesimal generator are polynomials in y. Affine transformations of the Ornstein-Uhlenbeck process, the Cox-Ingersoll-Ross process and the Jacobi process belong to the solutions of this stochastic differential equation family. Such processes exhibit specific patterns of the drift and volatility functions and can be represented by means of a basis of polynomial transforms which can be used to approximate the likelihood function. We also discuss the constraints on parameters to ensure the nonnegativity of the volatility function and the stationarity of the process. The possibility to fully characterize the dynamic properties of these processes explain why they are benchmark models for unconstrained variables such as asset returns (Ornstein-Uhlenbeck), for nonnegative variables as volatilities or interest rates (Cox, Ingersoll, Ross), or for variables which can be interpreted as probabilities (Jacobi).

Processus diffusions à fonctions propres polynomiales

RÉSUMÉ – Le but de cet article est de caractériser les processus diffusions univariés pour lesquels les fonctions propres du générateur infinitésimal sont des polynômes. Cette classe comprend essentiellement des transformés affines des processus de Ornstein-Uhlenbeck, de Cox-Ingersoll-Ross et de Jacobi. De tels processus correspondent à des choix particuliers des fonctions de translation et de volatilité, et définissent naturellement des bases de polynômes, qui peuvent servir pour construire des approximations de la fonction de vraisemblance. Nous décrivons de plus les conditions sur les paramètres nécessaires et suffisantes pour assurer la positivité de la volatilité et la stationnarité du processus. La possibilité d'étudier explicitement les propriétés dynamiques de ces processus explique qu'ils servent de modèles dynamiques de référence pour les variables sans contraintes de signe telles des rendements (Ornstein-Uhlenbeck), pour les variables positives, telles les volatilités ou les taux d'intérêt (Cox-Ingersoll-Ross) ou pour les variables s'interprétant comme des probabilités (Jacobi).

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1 Introduction

One dimensional stochastic differential equations (s.d.e), such as:

$$dy_t = \mu(y_t)dt + \sigma(y_t)dW_t \tag{1}$$

where (W_t) is a standard Brownian motion, μ and σ the drift and volatility functions, respectively, are basic specifications for describing the evolution in continuous time of financial returns (see e.g. BLACK, SCHOLES [1973]), interest rates (see e.g. VASICEK [1977], COX, INGERSOLL, ROSS [1985a, 1985b]), or macroeconomic series (see e.g., CAGETTI et alii [2002], CHEN, EPSTEIN [2002], ANDERSON, HANSEN, SARGENT [2003]). A recent literature points out the importance of spectral analysis of the associated infinitesimal generator of a stochastic differential equation. On the one hand, the knowledge of the spectral decomposition simplifies the computation of nonlinear predictions at any horizon. This feature is used for instance to determine the pattern of the term structure of interest rates when the short term interest rate follows a s.d.e. (PAGAN, HALL, MARTIN [1996]), or to derive pricing formulas in incomplete market framework (HANSEN, SCHEINKMAN [2003]). On the second hand, the spectral analysis of the infinitesimal generator underlies nonparametric estimation methods of drift and volatility functions. The basic idea is to estimate the infinitesimal generator either by a kernel approach (DAROLLES, FLORENS, GOURIÉROUX [2004]) or by projecting on a basis of polynomials[1], and to deduce from the first and second eigenfunctions the drift and volatility functions (see DEMOURA [1993]).

The aim of this paper is to characterize the one-dimensional stochastic differential equations, for which the eigenfunctions of the infinitesimal generator are polynomials in y. For these s.d.e., the estimation method by projection on polynomials will be accurate, even in finite sample (see KESSLER, SORENSEN [1999], LARSEN, SORENSEN [2003] for the asymptotic efficiency properties of this nonparametric projection approach).

The characterization of the diffusion processes with polynomial eigenfunctions is given in Section 2. We also provide in this section the eigenvalues, the expressions of the eigenfunctions, the stationarity conditions and the density of the marginal (stationary) distribution of the process. The proofs of the main results are gathered in Section 3, and Section 4 concludes.

2 Characterization

Let us consider a one dimensional stationary diffusion process:

$$dy_t = \mu(y_t)dt + \sigma(y_t)dW_t, \tag{1}$$

1. See DAROLLES, FLORENS, RENAULT [1997], HANSEN, SCHEINKMAN, TOUZI [1998], FLORENS, RENAULT, TOUZI [1998], DAROLLES, GOURIÉROUX [2001], CHEN, HANSEN, SCHEINKMAN [2005].

with drift and volatility functions denoted by μ and σ, respectively. Its infinitesimal generator A defined by:

$$A\psi(y) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[\psi(Y_{t+\Delta t}) - \psi(Y_t) \middle| Y_t = y\right], \tag{2}$$

explains how to compute the infinitesimal drift of the transformed series $(\psi(Y_t))$. The transformed series satisfies also a diffusion equation, and, by applying Ito's lemma, the generator A corresponds to the differential operator for second order differentiable functions ψ:

$$A\psi(y) = \mu(y)\frac{\partial\psi}{\partial y}(y) + \frac{1}{2}\sigma(y)^2 \frac{\partial^2\psi}{\partial y^2}(y). \tag{3}$$

This operator is generally self-adjoint, and admits a spectral decomposition with real nonpositive eigenvalues (see e.g. Hansen, Scheinkman [1995]). We assume[2] that:

Assumption A.1: *A is a compact operator, with distinct negative eigenvalues* λ_n, $n \in N$, *say, and eigenfunctions* ψ_n, $n \in N$.

The main result of the paper is described in the proposition below.

Proposition 1: *Under Assumption A1, the diffusion process admits polynomial eigenfunctions* ψ_n *with increasing degree n, if and only if one of the following conditions is satisfied:*

i) $\mu(y) = b(y - \beta)$, $\sigma^2(y) = c_0$, *where* $b < 0$, *and y is defined on R;*

ii) $\mu(y) = b(y - \beta)$, $\sigma^2(y) = c_1(y) + c_0$, *where* $b < 0$, *and y is defined on a semi-interval* $[- c_0/c_1, +\infty[$, *if* $c_1 > 0$, *or on a semi-interval* $]-\infty, -c_0/c_1]$, *if* $c_1 < 0$.

iii) $\mu(y) = b(y - \beta)$, $\sigma^2(y) = c_1(y - \gamma_1)(y - \gamma_2)$, *where* $b < 0$, $c < 0$, $\gamma_1 < \beta < \gamma_2$, *and y is defined on an interval* (γ_1, γ_2).

In any case the eigenvalues are:

$$\lambda_n = bn + \tfrac{1}{2}cn(n-1), \ n \geq 1, \text{ where } c = 0 \text{ for cases i) and ii).}$$

We get three types of processes which can be distinguished by their domain restrictions. They are affine transformations of the Ornstein-Uhlenbeck process, the Cox-Ingersoll-Ross process and the Jacobi process, respectively. The Ornstein-Uhlenbeck process, or mean-reverting process (see the negativity condition imposed on parameter b in i)) underlies the Vasicek model (see Vasicek [1977]). The Cox-Ingersoll-Ross process when $c_0 = 0$ in ii), and more generally the square

2. See Florens, Renault, Touzi [1998] for a discussion of this assumption.

root processes are used for describing the evolution of interest rates (see PAGAN, HALL, MARTIN [1996]), for deriving affine term structure models (DUFFIE, KAN [1996], DUFFIE, SINGLETON [1997]), or for defining time deformation[3]. Finally, the Jacobi process is appropriate for the evolution of a probability or a default rate, which is restricted between 0 and 1[4].

The proposition below provides different properties of these processes concerning their stationary distribution and their eigenfunctions. The stationary distribution belongs to the Pearson family, that is, the stationary density f satisfies $\frac{d \log f(y)}{dy} = \frac{ay+b}{cy^2+dy+c}$, say. This explains why, for a compact infinitesimal generator, the class of diffusion processes with polynomial eigenfunctions coincides with the class of stationary Markov processes with marginal distribution in the Pearson family (WONG [1964]). The expressions of the eigenfunctions are deduced from standard results on orthogonal polynomials (see e.g. ABRAMOWITZ, STEGUN [1965]). The eigenfunctions are not uniquely defined. The eigenfunctions given below, denoted ψ_n, are standardized with respect to the marginal (stationary) distribution of the process, that is, they satisfy $\int \psi_n(y)\psi_m(y) f(y) dy = 0$, if $n \neq m$, $= 1$, if $n = m$, where f is the p.d.f of the marginal distribution.

PROPOSITION 2:

i) The eigenfunctions of the Ornstein-Uhlenbeck process solution of the s.d.e.,

$$dy_t = b(y_t - \beta)dt + \sqrt{c_0} dW_t,$$

are the Hermite polynomials given by:

$$\tilde{He}_n(y_t) = (n!)^{1/2} \sum_{m=0}^{\left[\frac{n}{2}\right]} (-1)^m \frac{1}{m! 2^m (n-2m)!} \left(\frac{-2b}{c_0}\right)^{\frac{n-2m}{2}} (y_t - \beta)^{n-2m}.$$

They are standardized with respect to the marginal Gaussian distribution $N\left(\beta, -\frac{c_0}{2b}\right)$.

ii) The eigenfunctions of the square root process, solution of the s.d.e.,

$$dy_t = b(y_t - \beta)dt + \sqrt{c_1 y_t + c_0} dW_t,$$

3. See CONLEY, HANSEN, LUTTMER, SCHEINKMAN [1997], CARRASCO, HANSEN, CHEN [1999], GHYSELS, GOURIÉROUX, JASIAK [1995], GHYSELS, GOURIÉROUX, JASIAK [1998].

4. See NIELSEN, SAA REQUEJA, SANTA CLARA [1993], LANDO [1998], CAGETTI, HANSEN, SARGENT, WILLIAMS [2002] for such applications, LARSEN, SORENSEN [2003], GOURIEROUX, JASIAK [2006], for the use of Jacobi process.

are the Generalized Laguerre polynomials:

$$\tilde{L}_n^{(\alpha)}\left(-\frac{2b}{c_1}\left[y+c_0/c_1\right]\right)$$
$$=\binom{n+\alpha}{n}^{-1/2}\sum_{m=0}^{n}\left[\binom{n+\alpha}{n-m}\left(\frac{2b}{c_1}\right)\frac{\left(y+c_0/c_1\right)^m}{m!}\right]$$

where : $\alpha=-\frac{2c}{c_1}\left(\beta+c_0/c_1\right)-1$.

They are standardized with respect to the marginal distribution of the process, which is an affine transformation of the gamma distribution $\gamma\left[-\frac{2b}{c_1}\left(\beta+\frac{c_0}{c_1}\right)\right]$.

iii) The eigenfunctions of the Jacobi process, solution of the s.d.e.,

$$dy_t=b\left(y_t-\beta\right)dt+\sqrt{c\left(y_t-\gamma_1\right)\left(y_t-\gamma_2\right)}dW_t,$$

are the Jacobi polynomials given by:

(5)
$$\tilde{P}_n^{(\tilde{\alpha},\tilde{\beta})}\left(y_t\right)=\left[\frac{\Gamma\left(\tilde{\alpha}+n+1\right)\left(2n+\tilde{\alpha}+\tilde{\beta}+1\right)\Gamma\left(\tilde{\alpha}+1\right)\Gamma\left(\tilde{\beta}+1\right)}{n!\Gamma\left(\tilde{\alpha}+\tilde{\beta}+n+1\right)\Gamma\left(\tilde{\alpha}+\tilde{\beta}+2\right)\Gamma\left(\tilde{\beta}+n+1\right)}\right]^{1/2}$$
$$\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma\left(\tilde{\alpha}+\tilde{\beta}+n+m+1\right)}{\Gamma\left(\tilde{\alpha}+m+1\right)}\frac{\left(y_t-\gamma_2\right)^m}{\left(\gamma_2-\gamma_1\right)^m},$$

with

$$\tilde{\alpha}=\frac{2b}{c}\frac{\gamma_2-\beta}{\gamma_2-\gamma_1}-1,\tilde{\beta}=\frac{2b}{c}\frac{\beta-\gamma_1}{\gamma_2-\gamma_1}-1.$$

They are standardized with respect to the marginal distribution of the process, which is an affine transformation of the beta distribution $B\left(\tilde{\beta}+1,\tilde{\alpha}+1\right)$.

Under the restriction on parameters *b* and *c*, the eigenvalues λ_n,n ≥ 1, are strictly negative, in decreasing order. The negativity is a condition for the stationarity of the processes. Moreover the fact that the sequence is decreasing allows for an interpretation of the spectral decomposition in terms of nonlinear canonical analysis. More precisely, let us consider the optimization problem:

$$\max_{g,h} Corr\left[g\left(y_t\right),h\left(y_{t-1}\right)\right].$$

This problem admits the solution $g(y_t) = \psi_1(y_t), h(y_{t-1}) = \psi_1(y_{t-1})$ (up to affine transformations) and the optimal value of the objective function is $\exp\lambda_1$. Thus, ψ_1 defines the so-called first-order canonical direction, whereas $\exp\lambda_1$ is the first-order canonical correlation. Similarly, $\psi_n, \exp\lambda_n$ are the canonical direction and canonical correlation at order n, respectively.

3 Proof of the properties

The proof involves five steps. We first establish the necessary patterns of the drift and volatility functions, then the necessary expressions of the eigenvalues. In the third step, we explicit the constraints on parameters to ensure a nonnegative volatility and a stationary solution. In the fourth step, the marginal distributions of the processes are derived. Finally, we determine the standardized polynomial eigenfunctions.

3.1 The pattern of the drift and volatility functions

LEMMA 1: *If the eigenfunctions are polynomials, the drift is a polynomial of degree one:*

$$\mu(y) = b(y - \beta), \textit{ with } b < 0,$$

whereas the volatility is a polynomial of degree at most 2:

$$\sigma^2(y) = cy^2 + c_1 y + c_0.$$

Proof: It is known from (a generalized version of) Sturm-Liouville theorem, that the eigenfunctions of the infinitesimal generator satisfy the following shape restrictions (see e.g. CHEN, HANSEN and SCHEINKMAN [2005]):

- ψ_j crosses the x-axis precisely j times;
- its derivative ψ'_j admits precisely $j - 1$ distinct interior zeros (and the same sign between any consecutive zeros).

According to the shape restrictions, the first eigenfunction crosses the x-axis once, the second eigenfunction twice, and so forth. The first two polynomial eigenfunctions are of the form:

$$\psi_1(y) = y + a_{10},\ \psi_2(y) = y^2 + a_{21}y + a_{20}, \text{ say.}$$

They satisfy the condition:

$$A\psi_n(y) = \lambda_n \psi_n(y),\ n = 1,2 \text{ with } \lambda_n < 0,$$

$$\Leftrightarrow \mu(y)\frac{\partial\psi_n}{\partial y}(y)+\frac{1}{2}\sigma^2(y)\frac{\partial^2\psi_n}{\partial y^2}(y)=\lambda_n\psi_n(y) \quad n=1,2,$$

$$\Leftrightarrow \begin{cases}\mu(y)=\lambda_1(y+a_{10})\\ \mu(y)(2y+a_{21})+\sigma^2(y)=\lambda_2\left(y^2+a_{21}y+a_{20}\right).\end{cases}$$

Lemma 1 is deduced by solving this system with respect to the drift and volatility values.

Q.E.D.

3.2 Expression of the eigenvalues

LEMMA 2 *If the eigenfunctions are polynomials, the eigenvalues are:* $\lambda_n = nb+\frac{1}{2}cn(n-1)$, *where* $c \le 0$, *and* $b < 0$, *if* $c = 0$.

Proof: By replacing μ and σ^2 by their expressions, the spectral condition $A\psi_n(y) = \lambda_n\psi_n(y)$ becomes:

$$b(y-\beta)\frac{\partial\psi_n}{\partial y}(y)+\frac{1}{2}\left(cy^2+c_1y+c_0\right)\frac{\partial^2\psi_n}{\partial y^2}(y)=\lambda_n\psi_n(y).$$

When $\psi_n(y)=y^n+a_{n,n-1}y^{n-1}+\ldots+a_{n0}$ is a polynomial, we get by identifying the coefficients of the terms of degree n:

$$nb+\frac{1}{2}cn(n-1)=\lambda_n.$$

For large n, λ_n is equivalent to either $\frac{1}{2}cn(n-1)$, if $c \neq 0$, or nb, if $c = 0$. We deduce the constraints on parameters b and c to ensure that λ_n is negative. **Q.E.D.**

3.3 The constraints on the parameters

case i): **Constant volatility.**

The volatility is $\sigma^2(y) = c_0 > 0$ and the eigenvalues are $\lambda_n = nb$, with $b < 0$. These constraints are sufficient to characterize affine transformations of the Ornstein-Uhlenbeck process.

case ii): **Affine volatility.**

The volatility is $\sigma^2(y) = c_1y + c_0$, with $c_1 \neq 0$ and the eigenvalues are $\lambda_n = nb$, with $b < 0$. The positivity of the volatility is ensured if the domain of admissible values of y is restricted:

$$y\in]-c_0/c_1,+\infty[,\ if\ c_1>0\ ,y\in]-\infty,-c_0/c_1[,\ if\ c_1<0.$$

These constraints are sufficient to characterize affine transformations of the square root process.

case iii): **Quadratic volatility**.

The volatility is $\sigma^2(y) = cy^2 + c_1 y + c_0$ and the eigenvalues are $\lambda_n = nb + \frac{1}{2}cn(n-1)$, with $c < 0$. Since $c < 0$, the volatility function can take positive values, if and only, if the polynomial $\sigma^2(y) = cy^2 + c_1 y + c_0$ has two distinct real roots $\gamma_1 < \gamma_2$.

LEMMA 3: β *is between the roots* γ_1 *and* γ_2.

Proof:

The strict positivity of the volatility $\sigma^2(y) = cy^2 + c_1 y + c_0$ implies that $y \in (\gamma_1,\gamma_2)$, which implies that $E(y) = \beta$ belongs to the (γ_1,γ_2) interval. **Q.E.D**

Then, it can be checked that the process is well-defined, stationary, with range (γ_1,γ_2) (see GOURIEROUX, JASIAK [2006] for a complete proof].

3.4 Stationary distributions

It is known that the density function of the stationary distribution of a diffusion process is proportional to:

(1) $$f(x) \sim \frac{1}{\sigma^2(x)} \exp\left[2\int_a^x \frac{\mu(y)}{\sigma^2(y)} dy\right],$$

where a is an arbitrary interior point of the state space.

case i): **Ornstein-Uhlenbeck process**.

The drift and volatility functions are $\mu(y) = b(y-\beta)$, and $\sigma^2(y) = c_0$, which yields the p.d.f of the stationary distribution proportional to:

(2) $$\frac{1}{c_0}\exp[\frac{b}{c_0}(y-\beta)^2].$$

Therefore, we get a normal distribution $N\left(\beta, -\frac{c_0}{2b}\right)$ with p.d.f:

(3) $$f(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{-\frac{c_0}{2b}}} \exp\left[\frac{b}{c_o}(y-\beta)^2\right].$$

case ii): **Square root process**.

For the square root process, we have $\mu(y) = b(y-\beta)$ and $\sigma^2(y) = c_1 y + c_0$. Thus, the p.d.f of the stationary distribution is proportional to:

(4) $$\frac{1}{c_1}\left(x + \frac{c_0}{c_1}\right)^{-\frac{2b}{c_1}\left(\beta + \frac{c_0}{c_1}\right) - 1} \exp\left(\frac{2b}{c_1}x\right).$$

Let us consider the case $c_1 > 0$, which corresponds to the domain $\left[-\frac{c_0}{c_1},+\infty\right]$ for the process. Thus, the p.d.f of the stationary distribution corresponds to a gamma distribution with drift:

(5)

$$f(y)=\frac{\left(-\frac{2b}{c_1}\right)^{-\frac{2b}{c_1}\left(\beta+\frac{c_0}{c_1}\right)}}{\Gamma\left(-\frac{2b}{c_1}\left(\beta+\frac{c_0}{c_1}\right)\right)}\exp\left[-\left(-\frac{2b}{c_1}\right)\left(y+\frac{c_0}{c_1}\right)\right]\left(y+\frac{c_0}{c_1}\right)^{-\frac{2b}{c_1}\left(\beta+\frac{c_0}{c_1}\right)-1} I_{\left(-\frac{c_0}{c_1},+\infty\right)}(y),$$

where $\mathbb{1}_{\left(-\frac{c_0}{c_1},+\infty\right)}$ denotes the indicator function of the semi-interval.

Thus, $y_t = \theta_0 z_t + \theta_1$, where $\theta_0 = -\frac{c_1}{2b}$, $\theta_1 = -\frac{c_0}{c_1}$ and z_t follows the gamma distribution with parameter $-\frac{2b}{c_1}\left(\beta+\frac{c_0}{c_1}\right)$.

case iii): **Jacobi process**.

The drift and volatility functions are defined by $\mu(y) = b(y-\beta)$ and $\sigma^2(y) = c(y-\gamma_1)(y-\gamma_2)$ with $c < 0$. The p.d.f of the stationary distribution is proportional to:

(6) $$\frac{1}{-c}(y-\gamma_1)^{\frac{2b}{c}\frac{\beta-\gamma_1}{\gamma_2-\gamma_1}-1}(\gamma_2-y)^{\frac{2b}{c}\frac{\gamma_2-\beta}{\gamma_2-\gamma_1}-1}.$$

We deduce that the p.d.f :

(7) $$f(y)=\frac{(y-\gamma_1)^{\frac{2b}{c}\frac{\beta-\gamma_1}{\gamma_2-\gamma_1}-1}(\gamma_2-y)^{\frac{2b}{c}\frac{\gamma_2-\beta}{\gamma_2-\gamma_1}-1}}{(\gamma_2-\gamma_1)^{\frac{2b}{c}-1}B\left(\frac{2b}{c}\frac{\beta-\gamma_1}{\gamma_2-\gamma_1},\frac{2b}{c}\frac{\gamma_2-\beta}{\gamma_2-\gamma_1}\right)}1_{(\gamma_1,\gamma_2)}(y)$$

corresponds to a Beta distribution defined on the interval $[\gamma_1,\gamma_2]$.

3.5 Polynomial eigenfunctions

It is easily checked that the differential equation:

(8) $$\frac{1}{2}\left(cy^2+c_1y+c_0\right)\frac{\partial^2\psi_n}{\partial y^2}(y)+b(y-\beta)\frac{\partial\psi_n}{\partial y}(y)=\left[nb+\frac{1}{2}cn(n-1)\right]\psi_n(y),$$

admits a solution which is a polynomial of degree n. Therefore, there is a basis of canonical eigenfunctions corresponding to polynomials of increasing degrees. We have just to explicit the solutions for the three cases described in Proposition 1.

case i): **Ornstein-Uhlenbeck process**.
The differential equation (3.8) with c = 0, and $c_1 = 0$, that is,

(9) $$b(y-\beta)\frac{\partial\psi_n}{\partial y}+\frac{1}{2}c_0\frac{\partial^2\psi_n}{\partial y^2}=bn\psi_n(y),$$

is directly related to the so-called Hermite equation:

(10) $$\frac{\partial^2\Phi}{\partial z^2}(z)+(-z)\frac{\partial\Phi(z)}{\partial z}+n\Phi(z)=0,$$

after an appropriate change of variable. Indeed, let us consider an affine transformation : $y = \alpha z + \gamma$, and denote : $\Phi_n(z)=\psi_n(\alpha z+\gamma)$. After substituting $\alpha z + \gamma$ for y, equation (3.9) can be written as:

(11) $$\frac{c_o}{2\alpha^2}\frac{\partial^2\Phi_n(z)}{\partial z^2}+\frac{b}{\alpha}(\alpha z+\gamma-\beta)\frac{\partial\Phi_n(z)}{\partial z}-bn\Phi_n(z)=0\,.$$

By choosing $\alpha = \sqrt{\frac{-c_o}{2b}}$, we get:

$$\frac{\partial^2\Phi_n(z)}{\partial z^2}+\left(-z+\frac{\beta-\gamma}{\alpha}\right)\frac{\partial\Phi_n(z)}{\partial z}+n\Phi_n(z)=0,$$

and the equation reduces to the Hermite equation when $\beta = \gamma$. A solution Φ_n of the Hermite equation (3.10) is the Hermite polynomial (see ABRAMOWITZ and STEGUN [1965], p.775, formula 22.3.11):

(12) $$\begin{aligned}\Phi_n(z)&=He_n(z)\\&=n!\sum_{m=0}^{\left[\frac{n}{2}\right]}(-1)^m\frac{1}{m!2^m(n-2m)!}z^{n-2m}.\end{aligned}$$

Therefore, the solutions of equation (3.9) are the Hermite polynomials:

(13)
$$\tilde{He}_n(y_t)=He_n\left[\sqrt{\frac{-2b}{c_o}}(y_t-\beta)\right]n!\sum_{m=0}^{\left[\frac{n}{2}\right]}(-1)^m\frac{1}{m!2^m(n-2m)!}\left(\frac{-2b}{c_0}\right)^{\frac{n-2m}{2}}(y_t-\beta)^{n-2m}.$$

Finally, the Hermite polynomials have to be standardized. It is known that the basic Hermite polynomials He_n are orthogonal with respect to the standard normal distribution and such that (see ABRAMOWITZ and STEGUN [1965], p.775, formula 22.2.15) :

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{z_t^2}{2}\right) He_n(z_t)^2 \, dz_t = \sqrt{2\pi} n!$$

or,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{z_t^2}{2}\right) He_n(z_t)^2 \, dz_t = n!.$$

Therefore,

$$\begin{aligned} He_n^*(z_t) &= \frac{He_n(z_t)}{(n!)^{1/2}} \\ &= (n!)^{1/2} \sum_{m=0}^{\left[\frac{n}{2}\right]} (-1)^m \frac{1}{m! 2^m (n-2m)!} z_t^{n-2m}, \end{aligned} \tag{14}$$

defines an orthonormal sequence of polynomials with respect to the standard normal distribution of z_t. Thus,

$$\tilde{H}^* e_n(y_t) = (n!)^{1/2} \sum_{m=0}^{\left[\frac{n}{2}\right]} (-1)^m \frac{1}{m! 2^m (n-2m)!} \left(\frac{-2b}{c_0}\right)^{\frac{n-2m}{2}} (y_t - \beta)^{n-2m}, \tag{15}$$

defines an orthonormal sequence with respect to the distribution $N\left(\beta; \frac{c_o}{2b}\right)$.

case ii): **Square root process**.

The differential equation is:

$$\frac{1}{2}(c_1 y + c_o) \frac{\partial^2 \psi_n}{\partial y^2}(y) + b(y-\beta) \frac{\partial \psi_n}{\partial y}(y) = nb\psi_n(y). \tag{16}$$

Let us introduce the transformed variable:

$$z_t = \frac{-2b}{c_1}(y_t + c_0/c_1), \tilde{\psi}\left[\frac{-2b}{c_1}(y_t + c_0/c_1)\right] = \psi(y).$$

The differential equation becomes:

$$z \frac{\partial^2 \tilde{\psi}}{\partial z^2}(z) + \left(-\frac{2b}{c_1}\left(\beta + \frac{c_0}{c_1}\right) - z\right) \frac{\partial \tilde{\psi}}{\partial z}(z) + n\tilde{\psi}(z) = 0. \tag{17}$$

This is the Kummer's equation, which admits as solution the Generalized Laguerre polynomials (see ABRAMOWITZ-STEGUN [1965], formulas 22.6.15 and 22.3.9). These polynomials are given by :

$$L_n^{(\alpha)}(z)=\sum_{m=0}^{n}\left[(-1)^m\binom{n+\alpha}{n-m}\frac{z^m}{m!}\right],$$

where $\alpha=-\frac{2b}{c_1}\left(\beta+\frac{c_0}{c_1}\right)-1$. These polynomials are orthogonal with respect to the gamma distribution with parameter $\alpha + 1$:

$$\int_0^{\infty} L_z^{(\alpha)}(z)L_m^{(\alpha)}(z)\frac{1}{\Gamma(\alpha+1)}\exp(-z)z^{\alpha}dz=0,\forall n\neq m.$$

Moreover, they satisfy (ABRAMOWITZ-STEGUN [1965], formula 22.2-12):

$$\int_0^{\infty}\left[L_n^{(\alpha)}(z)\right]^2\frac{1}{\Gamma(\alpha+1)}\exp(-z)z^{\alpha}dz=\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)n!}=\binom{n+\alpha}{n},\forall n.$$

Thus, an orthonormal basis of polynomials is:

$$\begin{aligned}\tilde{L}_n^{(\alpha)}(z)&=\binom{n+\alpha}{n}^{-1/2}L_n^{\alpha}(z)\\&=\binom{n+\alpha}{n}^{-1/2}\sum_{m=0}^{n}\left[(-1)^m\binom{n+\alpha}{n-m}\frac{z^m}{m!}\right].\end{aligned}$$

case iii): **Jacobi process**.

After introducing the roots γ_1, γ_2, the equation (3.8) can be written as:

(18)

$$[-y^2+(\gamma_1+\gamma_2)y-\gamma_1\gamma_2]\frac{\partial^2\psi}{\partial y^2}(y)-\frac{2b}{c}(y-\beta)\frac{\partial\psi}{\partial y}(y)+\frac{2}{c}\left[nb+\frac{1}{2}cn(n-1)\right]\psi(y)=0.$$

i) Let us first assume $[\gamma_1,\gamma_2]=[-1,1]$; the equation (3.18) becomes:

$$(1-y^2)\frac{\partial^2\psi}{\partial y^2}(y)+\left(\frac{2b}{c}\beta-\frac{2b}{c}y\right)\frac{\partial\psi}{\partial y}(y)+\frac{2}{c}\left[nb+\frac{1}{2}cn(n-1)\right]\psi(y)=0, \quad (19)$$

It is known (see ABRAMOWITZ and STEGUN [1965], p.781, formula 22.6.1) that this differential equation admits as solutions the Jacobi polynomials (see ABRAMOWITZ and STEGUN [1965], p.775, formula 22.3.2):

(20) $$P_n^{(\tilde{\alpha}^*,\tilde{\beta}^*)}(y_t^*) = \frac{\Gamma\left(\tilde{\alpha}^*+n+1\right)}{n!\Gamma\left(\tilde{\alpha}^*+\tilde{\beta}^*+n+1\right)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma\left(\tilde{\alpha}^*+\tilde{\beta}^*+n+m+1\right)}{2^m\Gamma\left(\tilde{\alpha}^*+m+1\right)}\left(y_t^*-1\right)^m,$$

with $\tilde{\alpha}^* = \frac{b^*}{c^*}\left(1-\beta^*\right)-1$ and $\tilde{\beta}^* = \frac{b^*}{c^*}\left(\beta^*+1\right)-1$. Moreover these polynomials satisfy the following inner product conditions (see ABRAMOWITZ and STEGUN [1965], p.773, formulas 22.1.1, 22.1.2, and 22.2.1):

$$\int_{-1}^{1}\left(1-y_t^*\right)^{\tilde{\alpha}^*}\left(1+y_t^*\right)^{\tilde{\beta}^*}P_n\left(y_t^*\right)P_m\left(y_t^*\right)dy_t^* = 0, \quad n \neq m,$$

and

(21)

$$\int_{-1}^{1}\left(1-y_t^*\right)^{\tilde{\alpha}^*}\left(1+y_t^*\right)^{\tilde{\beta}^*}P_n^2\left(y_t^*\right)dy_t^* = \frac{2^{\tilde{\alpha}^*+\tilde{\beta}^*+1}}{2n+\tilde{\alpha}^*+\tilde{\beta}^*+1}\frac{\Gamma\left(\tilde{\alpha}^*+n+1\right)\Gamma\left(\tilde{\beta}^*+n+1\right)}{n!\Gamma\left(\tilde{\alpha}^*+\tilde{\beta}^*+n+1\right)}.$$

Let us now standardize the weight function in order to recover the beta stationary distribution. We get:

$$\int_{-1}^{1}\left(1-y_t^*\right)^{\tilde{\alpha}^*}\left(1+y_t^*\right)^{\tilde{\beta}^*}dy_t^* = 2^{\tilde{\alpha}^*+\tilde{\beta}^*+1}\frac{\Gamma\left(\tilde{\alpha}^*+1\right)\Gamma\left(\tilde{\beta}^*+1\right)}{\Gamma\left(\tilde{\alpha}^*+\tilde{\beta}^*+2\right)}$$

After this standardization, equation (3.21) becomes:

(22) $$\frac{\Gamma\left(\tilde{\alpha}^*+\tilde{\beta}^*+2\right)}{2^{\tilde{\alpha}^*+\tilde{\beta}^*+1}\Gamma\left(\tilde{\alpha}^*+1\right)\Gamma\left(\tilde{\beta}^*+1\right)}\int_{-1}^{1}\left(1-y_t^*\right)^{\tilde{\alpha}^*}\left(1+y_t^*\right)^{\tilde{\beta}^*}P_n^2\left(y_t^*\right)dy_t^*$$
$$= \frac{\Gamma\left(\tilde{\alpha}^*+\tilde{\beta}^*+2\right)\Gamma\left(\tilde{\alpha}^*+n+1\right)\Gamma\left(\tilde{\beta}^*+n+1\right)}{\left(2n+\tilde{\alpha}^*+\tilde{\beta}^*+1\right)\Gamma\left(\tilde{\alpha}^*+1\right)\Gamma\left(\tilde{\beta}^*+1\right)\Gamma\left(\tilde{\alpha}^*+\tilde{\beta}^*+n+1\right)n!}$$

which yields the standardized Jacobi polynomials:

(23) $$\tilde{P}_n^{(\tilde{\alpha}^*,\tilde{\beta}^*)}(y_t^*) = \left[\frac{\Gamma\left(\tilde{\alpha}^*+n+1\right)\left(2n+\tilde{\alpha}^*+\tilde{\beta}^*+1\right)\Gamma\left(\tilde{\alpha}^*+1\right)\Gamma\left(\tilde{\beta}^*+1\right)}{n!\Gamma\left(\tilde{\alpha}^*+\tilde{\beta}^*+n+1\right)\Gamma\left(\tilde{\alpha}^*+\tilde{\beta}^*+2\right)\Gamma\left(\tilde{\beta}^*+n+1\right)}\right]^{1/2}$$
$$\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma\left(\tilde{\alpha}^*+\tilde{\beta}^*+n+m+1\right)}{2^m\Gamma\left(\tilde{\alpha}^*+m+1\right)}\left(y_t^*-1\right)^m.$$

ii) Finally, from the stochastic differential equation on (–1,1):

$$dy_t^* = b^*\left(y_t^* - \beta^*\right)dt + \sqrt{c^*\left(y_t^*+1\right)\left(y_t^*-1\right)}dW_t^*,$$

we deduce the solution of the stochastic differential equation on (γ_1,γ_2):

$$dy_t = b\left(y_t - \beta\right)dt + \sqrt{c\left(y_t-\gamma_1\right)\left(y_t-\gamma_2\right)}dW_t$$

by applying the affine transform $y_t = \frac{\gamma_2-\gamma_1}{2}y_t^* + \frac{\gamma_2+\gamma_1}{2}$. We have $b = b^*$, $c = c^*$, and $\beta = \left(\gamma_1+\gamma_2\right)/2 + \beta^*\left(\gamma_1+\gamma_2\right)/2$.

The polynomial eigenfunctions of the general s.d.e. are derived by applying the same affine transformations to the Jacobi polynomials. We get:

(24)
$$\tilde{P}_n^{(\tilde{\alpha},\tilde{\beta})}\left(y_t\right) = \left[\frac{\Gamma\left(\tilde{\alpha}+n+1\right)\left(2n+\tilde{\alpha}+\tilde{\beta}+1\right)\Gamma\left(\tilde{\alpha}+1\right)\Gamma\left(\tilde{\beta}+1\right)}{n!\Gamma\left(\tilde{\alpha}+\tilde{\beta}+n+1\right)\Gamma\left(\tilde{\alpha}+\tilde{\beta}+2\right)\Gamma\left(\tilde{\beta}+n+1\right)}\right]^{1/2} \sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma\left(\tilde{\alpha}+\tilde{\beta}+n+m+1\right)}{\Gamma\left(\tilde{\alpha}+m+1\right)}\frac{\left(y_t-\gamma_2\right)^m}{\left(\gamma_2-\gamma_1\right)^m}.$$

with $\tilde{\alpha} = \frac{2b}{c}\frac{\gamma_2-\beta}{\gamma_2-\gamma_1} - 1$ and $\tilde{\beta} = \frac{2b}{c}\frac{\beta-\gamma_1}{\gamma_2-\gamma_1} - 1$.

4 Concluding remarks

The knowledge of the spectral decomposition of the infinitesimal generator is a main step for the analysis of nonlinear dynamics of diffusion processes. In this paper, we have characterized and discussed the diffusion processes with polynomial eigenfunctions, and shown that they are affine transformations of Ornstein-Uhlenbeck, Cox-Ingersoll-Ross, or Jacobi processes. The spectral decomposition is useful for different purposes. For instance, the conditional p.d.f. of y_t given y_{t-1} can be written as:

$$f\left(y_t \mid y_{t-1}\right) = f\left(y_t\right)\sum_{n=0}^{\infty}\exp\left(\lambda_n\right)\psi_n\left(y_t\right)\psi_n\left(y_{t-1}\right).$$

This series expansion can be used to approximate the ratio of the conditional p.d.f. and the marginal one by well-chosen cross polynomials in y_t, y_{t-1}. In particular, when the marginal p.d.f. has a simple expression whereas the conditional p.d.f. has no tractable form, the truncated expansion can be used to define approximate maximum likelihood method. This is typically the case for the Jacobi process.

Moreover, the spectral decomposition is also useful for computing nonlinear predictions at different horizons. More precisely, any function g can be decomposed on the spectral basis as:

$$g(y) = \sum_{n=0}^{\infty} < g, \psi_n > \psi_n(y),$$

where : $< g, \psi_n > = \int g(y)\psi_n f(y)dy$ denotes the inner product with respect to the marginal p.d.f. Then, the prediction of $g(y)$ at date t for horizon h is:

$$E\left[g(y_{t+h}) \mid y_t\right] = \sum_{n=0}^{\infty} \exp(h\lambda_n) < g, \psi_n > \psi_n(y_t).$$

This type of formula provides the dependence of the prediction with respect to the horizon. It is especially useful for studying term structures of interest rates or term structures of volatilities, or for analysing the multiplier effects in a nonlinear framework, i.e. the so-called impulse response functions. ■

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