ANNALES D'ÉCONOMIE ET DE STATISTIQUE. – N° 82 – 2006

On the survival and Irreductibility Assumptions for Financial Markets with Nominal Assets

Styliani KANELLOPOULOU*, Abdelkrim SEGHIR and Leila TRIKI*****

ABSTRACT. – We are interested in proving an equilibrium existence result in a general equilibrium model with incomplete nominal asset markets. When we relax the assumption of strict positivity of initial endowments, then, as it is the case for every general equilibrium existence problem, we need to introduce survival and irreducibility assumptions, whose formulation is the object of this paper. The financial economy that we consider is a two period exchange economy where agents' preferences on their consumption sets may be non-ordered and do not satisfy monotonicity.

Hypothèses de survie et d'irréductibilité dans le cadre de marchés financiers incomplets et actifs nominaux

RÉSUMÉ. – Nous nous intéressons à démontrer l'existence d'un équilibre dans une économie avec marchés incomplets et actifs nominaux, où les préférences des agents sur leurs ensembles de consommations peuvent être non-ordonnés et ne satisfont pas d'hypothèse de monotonie. Quand nous relaxons l'hypothèse de stricte positivité des ressources initiales, nous devons introduire des hypothèses de survivance et d'irréductibilité adéquate dont la formulation est le but de ce papier.

This paper is a substantial improvement of a working paper [9]. We are very grateful to Monique Florenzano and Pascal Gourdel for having helped us correcting an error in the preliminary version. We also thank Filipe Martins da Rocha for helpful comments.
* S. Kanellopoulou : DG-Research / Financial Research Division European Central Bank Kaiserstrasse 29 60311 Frankfurt am Main Germany. E-mail: styliani.kanellopoulou@univ-paris1.fr
** A. Seghir :The American University in Cairo, Economics Department, 113 Kasr El Aini St., P.O. Box 2511, Cairo, 11511, Egypt. E-mail: kseghir@aucegypt.edu
*** L. Triki : CERMSEM, Université Paris 1, 106-112 boulevard de l'Hopital, 75647 Paris Cedex 13. E-mail: triki@univ-paris1.fr

1 Introduction

In the Arrow-Debreu theory of general equilibrium, it is often assumed that every agent’s endowment lies in the interior of his consumption set. As well, in the incomplete markets literature, several authors proved the existence of equilibrium under this strong survival assumption. As Arrow and Debreu already stated, *this strong survival assumption is clearly unrealistic, and weakening is desirable*. However, boundary endowments may cause a lack of existence of equilibrium. In fact, one meets with the mathematical difficulty that budget constraints of an agent may not be lower semicontinuous when his wealth lies on the boundary of his consumption set. In a financial economy, as in an Arrow-Debreu economy, under standard assumptions but without strong survival, one obtains existence of a quasiequilibrium, a weaker notion of equilibrium, that we define in the paper for a financial economy with incomplete markets. Then, if markets are complete, the assumption that the aggregate endowment lies in the interior of the sum of consumption sets, joint to some continuity and irreducibility assumption, leads to the existence of an equilibrium. When markets are incomplete, several examples show that even in a well-behaved financial economy, equilibrium may not exit under this survival assumption (refer to [10]). The main concern of this paper is to define a survival assumption adapted to this case.

We deal here with a two period financial economy with nominal assets, and consumption at each period, defined under weak assumptions. Consumption sets coincide with the positive orthant of the commodity space, but agent’s preferences may be non ordered and we do not assume monotonicity. As assets are nominal, we have two kinds of results, depending on whether we look for the equilibrium existence, or whether, following Cass [1], we try to prove that every no-arbitrage price can be embedded in an equilibrium. The survival assumption used in the first result, adapted from Gottardi and Hens [4], is rather abstract. For a result *à la Cass*, a second survival assumption is directly related with the fundamentals of the economy. Taking into account that our definition of the economy is very general, when related to characteristics of the economy, the first survival assumption is not very much more restrictive than the second one.

A second concern of the paper is to define an irreducibility assumption for the financial economy. The irreducibility notion that we use is the one proposed by McKenzie [8], reformulated with incomplete markets environment in order to include the existing restrictions on income transfers across states, which may prevent some agents from being able to offer something valuable to some other agent. The irreducibility assumption, also adapted from Gottardi and Hens [4], introduced in this paper, is the same in both equilibrium existence results.

The paper is organized as follows. In the next section, we present the model, assumptions and results. Section 3 is devoted to the proof of the two equilibrium existence results, obtained in two major steps: proving the existence of a quasiequilibrium under general assumptions, and introducing adequate survival and irreducibility assumptions under which quasiequilibrium is achieved as an equilibrium. Finally, the last section of the paper concentrates on a discussion of the survival and irreducibility notions, relating them with the fundamentals of the economy.

2 The model and results

Let an economy extend over two time periods $t = 0, 1$ with a finite set S of possible states of nature $s \in S$ in period 1. For convenience, we assume that $0 \notin S$ and $s = 0$ denotes the state of the world (known with certainty) at period 0. We denote $\overline{S} = \{0\} \cup S$. There is a finite set L of commodities for consumption at date 0 and in each state at date 1. There is a finite set J of nominal assets for trade at date 0. Asset $j \in J$ yields $r_j(s)$ units of account in state s at date 1; let $R = (r_j(s))_{(s,j)\in S\times J}$ denote the $S \times J$ - real matrix of returns. There is a finite set I of consumers (or agents) in the economy. Each consumer has a consumption set $X^i \subset \mathbb{R}^{L\overline{S}}$, an initial endowment vector $\omega^i\ in \mathbb{R}^{L\overline{S}}$, a preference correspondence $P^i\ :\ \prod_{i\in I} X^i \to X^i$ describing the consumption plans in X^i which are strictly preferred to x^i $\left(x^i \notin P^i(x)\right)$ by consumer i, taking as given the consumption $((x^{i'})_{i'\neq i})$ of the other agents, and a portfolio space Z^i which describes the portfolios available for consumer i. The collection :

$$\varepsilon = \left(\left(X^i, P^i, \omega^i, Z^i\right)_{i\in I}, R\right)$$

summarizes the financial economy. Vectors of commodity and asset prices will be denoted $p \in \mathbb{R}^{L\overline{S}}$ and $q \in \mathbb{R}^J$, respectively. The bundle (q, R) will designate the financial market of the economy. We adopt the compact notations:

- $X = \prod_{i\in I} X^i$ and $Z = \prod_{i\in I} Z^i$;
- For each $x = (x_{\ell,s})_{(\ell,s)\in L\times\overline{S}}$ and $p = (p_{\ell,s})_{(\ell,s)\in L\times\overline{S}}$ in $\mathbb{R}^{L\overline{S}}$, we set for $s \in \overline{S}$, $x(s) = (x_{\ell,s})_{\ell\in L}$ in $\mathbb{R}^L$, and $p(s) = (p_{\ell,s})_{\ell\in L}$ in $\mathbb{R}^L$;
- For each $x, p \in \mathbb{R}^{L\overline{S}}$, $\overline{p}x$ denotes the vector $\left(p(s)\cdot x(s)\right)_{s\in\overline{S}}$ in $\mathbb{R}^{\overline{S}}$, where $p(s)\cdot x(s) = \sum_{\ell} p_{s,\ell}x_{s,\ell}$; and $\overline{p_1}x$ denotes the vector $\left(p(s)\cdot x(s)\right)_{s\in S}$ in $\mathbb{R}^S$;
- For each $z = (z_j)_{j\in J} \in \mathbb{R}^J$, Rz denotes the vector $(R(s)\cdot z)_{s\in S}$ in $\mathbb{R}^S$, where $R(s)$ denotes the sth row of matrix R and $R(s)\cdot z$ in $\mathbb{R}$ denotes $\sum_j r_{s,j}z_j$;
- For each $z, q \in \mathbb{R}^J$, $W(q)z$ denotes the vector $(-q\cdot z, Rz)$ in $\mathbb{R}^{\overline{S}}$, $W_0(q)\cdot z$ denotes $-q\cdot z$ and for every $s \in S$, $W_s(q)\cdot z$ denotes $R(s)\cdot z$;
- For each $x,\ y \in \mathbb{R}^{L\overline{S}}$, $x \geq y \Rightarrow x^k \geq y^k$, $\forall k \in L\times\overline{S}$.
- For each $x,\ y \in \mathbb{R}^{L\overline{S}}$, $x \neq y$, and $x \gg y \Rightarrow x^k > y^k$, $\forall k \in L\times\overline{S}$.
- For each $x^i \in \mathbb{R}^{L\overline{S}}$, $i \in I$, the vector $x^i_{-s} \in \mathbb{R}^{LS}$ denotes the vector x^i to which we retrieved component $x^i(s)$. With this notation, $x^i = (x^i(s), x^i_{-s})$.

Then, given a commodity price vector $p \in \mathbb{R}^{L\overline{S}}$ and an asset price vector $q \in \mathbb{R}^{J}$ measured in units of account, the budget set of consumer *i* is:

$$B^i(p,q) = \left\{ \left(x^i, z^i\right) \in X^i \times Z^i \,\middle|\, \overline{p}\left(x^i - \omega^i\right) \leq W(q) z^i \right\}.$$

Given this pair (p,q) of commodity and asset price vectors, each agent *i* solves the following optimization problem:

$$(M^i) \quad : \begin{cases} \text{Find} & (x^i, z^i) \in B^i(p,q) \\ \text{such that} & \left(P^i(x) \times Z^i\right) \cap B^i(p,q) = \emptyset. \end{cases}$$

Definition 2.1: *A financial equilibrium is a collection of prices and actions* $(\overline{p}, \overline{q}, \overline{x}, \overline{z}) \in \mathbb{R}^{L\overline{S}} \times \mathbb{R}^J \times X \times Z$, *such that* $\forall s \in \overline{S}$, $\overline{p}(s) \neq 0$, *and:*

(1) *Given* $(\overline{p}, \overline{q}) \in \mathbb{R}^{L\overline{S}} \times \mathbb{R}^J$, $\forall i \in I$, $(\overline{x}^i, \overline{z}^i)$ *solves agent i's decision problem* (M^i),

(2) $\sum_{i \in I} \overline{x}^i = \sum_{i \in I} \omega^i$,

(3) $\sum_{i \in I} \overline{z}^i = 0.$

Classically, (1) means that $(\overline{x}^i, \overline{z}^i)$ *is an optimal budget feasible plan for agent* $i \in I$, *given* $(\overline{p}, \overline{q})$. *Conditions (2) and (3) are market clearing conditions and ensure that agents' decisions are compatible.*

2.1 Standard Assumptions

Let us denote by $\widehat{X}$ the set of all attainable consumption allocations:

$$\widehat{X} := \left\{ x \in X : \sum_{i \in I} \left(x^i - \omega^i\right) = 0 \right\}.$$

We will maintain in this paper the following standard assumptions.

On the consumption side, we assume:

C1 For every $i \in I$, $X^i = \mathbb{R}_+^{L\bar{S}}$, $\omega^i \in X^i$ (i.e. $\omega^i \geq 0$), and the preference correspondence, $P^i : X \to X^i$, has an open graph and has convex values in X^i. Moreover, $x^i \notin P^i(x)$ and the preference correspondence satisfies an additional convexity property: $[y^i \in P^i(x)$ and $0 < \lambda \leq 1]$ imply $[x^i + \lambda(y^i - x^i) \in P^i(x^i)]$.

C2 For every $x \in \widehat{X}$, for every $i \in I$, for every $s \in \bar{S}$, there exists a consumption bundle $y^i \in X^i$, differing from x^i only at state s, such that $y^i \in P^i(x)$.

Note that the additional convexity property of P^i, $i \in I$, implies that we can choose $y^i(s)$ as close as we want from $x^i(s)$.

Assumption **C2** is an hypothesis of nonsatiation at every date-event pair and at every component of an attainable consumption allocation which can be reinforced to the existence of a desirable direction $e \in \mathbb{R}_+^L$:

C2' There is some $e \in \mathbb{R}_+^L$ such that for every $x \in \widehat{X}$, for every $s \in \bar{S}$, for some $\lambda > 0$, $(x^i(s) + \lambda e, x_{-s}^i) \in P^i(x)$ $\forall i \in I$.

Assumption **C2'** can itself be reinforced to the strict monotonicity of preference correspondences P^i at each component of $x \in \widehat{X}$:

C2" For every $x \in \widehat{X}$, for every $i \in I$, for every $s \in \bar{S}$, for every $y^i \in \mathbb{R}^{L\bar{S}}$, $y^i > 0$, one has $x^i + y^i \in P^i(x)$.

On the financial side, we assume:

F1 Assets are not redundant, i.e., the rank of the $S \times J$ -real matrix R is J. Moreover, there exists $\hat{z} \in \mathbb{R}^J$ satisfying $R\hat{z} \gg 0$.[1]

F2 For every $i \in I$, $Z^i = \mathbb{R}^J$.

2.2 No-arbitrage Condition

Let us just remind here the well-known results of no-arbitrage asset prices.

DEFINITION 2.2: *Given the return matrix R, asset prices $q \in \mathbb{R}^J$ do not offer any arbitrage opportunity if and only if* $\nexists z \in \mathbb{R}^J$ *such that* $W(q)z > 0$.

1. This assumption is satisfied if we assume, as in Werner [10], that $R \geq 0$ and that for every $s \in S$, $R(s) > 0$.

The following proposition states that at equilibrium, the financial market (q, R) does not offer any arbitrage opportunity to any agent.

PROPOSITION 2.1: *Let* ε *be an economy with financial markets satisfying the standard assumptions and let* $(\overline{p}, \overline{q}, \overline{x}, \overline{z})$ *be an equilibrium of* ε. *Then the financial market* (q, R) *does not offer any arbitrage opportunity to any agent.*

We also recall the following lemma that enables us to characterize a no-arbitrage asset price vector as a strictly positive linear combination of the returns.

LEMMA 2.2: *An asset price vector* $q \in \mathbb{R}^J$ *is a no-arbitrage asset price if and only if there exists* $\lambda \in \mathbb{R}^{\overline{S}}_{++}$ *such that* $\lambda_0 = 1$, *and* $\sum_{s \in S} \lambda_s R(s)$.

It is well known that $\lambda \in \mathbb{R}^{\overline{S}}_{++}$ is unique if and only if $J = S$, that is, if and only if markets are complete.

DEFINITION 2.3: *We denote by* $\mathbb{Q}$ *the set of no-arbitrage price vectors, that is to say:*

$$\mathbb{Q} := \left\{ q \in \mathbb{R}^J \mid q = \sum_{s \in S} \lambda_s R(s), \text{ for some } \lambda \in \mathbb{R}^{\overline{S}}_{++} \right\}.$$

2.3 Survival and irreducibility assumptions

As in a classical general equilibrium model, the weak survival assumption that $\omega^i \in X^i$ for each $i \in I$ is not sufficient for existence of equilibrium. Moreover, as several examples show, under the usual strong survival assumption that $\sum_{i \in I} \omega^i \gg 0$, equilibrium may not exist, even in a well-behaved financial economy.[2] In fact, under the standard assumptions, we obtain existence of a quasiequilibrium, a weaker notion of equilibrium that we shall define in Section 3. In order to prove the existence of an equilibrium, we set a specific strong survival assumption and introduce an irreducibility assumption.

The survival assumption depends on the equilibrium existence result we address (existence of a financial equilibrium, or existence of equilibrium, given a no-arbitrage asset price vector).

For the existence of an equilibrium, the survival assumption is adapted from Gottardi-Hens [4]. We posit:

SS1 For all $(p, q) \in \mathbb{R}^{L\overline{S}} \times \mathbb{R}^J$, $p(s) \neq 0$, $\forall s \in S$, and $q \in \text{cl}\, \mathbb{Q}$, with $(p(0), q) \neq (0, 0)$, there is some $i \in I$, some $(x^i, z^i) \in X^i \times Z^i$ such that $p(0) \cdot (x^i(0) - \omega^i(0)) + q \cdot z^i < 0$ and $\overline{p}_1(x^i - \omega^i) \leq R z^i$.

2. An example can be found in Werner [10].

In view of Assumption **F1**, Assumption **SS1** is equivalent to the existence, for each p such that $p(s) \neq 0,\ \forall s \in S$ and $q \in \mathrm{cl}\,\mathbb{Q}$, of an agent i (dependent on p and q) and a pair $(x^i, z^i) \in X^i \times Z^i$ such that:

$$\overline{p}(x^i - \omega^i) \ll W(q)z^i.$$

Assumptions **SS1** then appears as a natural adaptation to the incomplete market setting of the strong survival assumption made for obtaining existence of equilibrium in a complete market setting. At prices (p, q), at least one agent can consume satisfying strictly all his budget constraints. This (somewhat abstract) assumption will be related with the fundamentals of the model in the last section of the paper.

For the existence of equilibrium given a no-arbitrage asset price vector (which gives a complete description of the set of equilibrium asset price vectors), one could be tempted to introduce the following assumption:

SS2 For all $p \in \mathbb{R}^{L\overline{S}},\ p \neq 0$, there is some $i \in I$, some $(x^i, z^i) \in X^i \times Z^i$ such that $p(0)\cdot(x^i(0) - \omega^i(0)) + q \cdot z^i < 0$ and $\overline{p}_1(x^i - \omega^i) \leq Rz^i$.

We will see in the appendix that such an assumption implies that markets are complete. In order to get an equilibrium existence result in which, as Cass, we have a complete description of the set of asset equilibrium price vectors, we posit:

C3 There exists an agent $i \in I,\ \omega^i \gg 0$.

The irreducibility assumption, also adapted from Gottardi and Hens [4], is common to both equilibrium existence results:

IR For every non trivial partition $\{I_1, I_2\}$ of I, for all $x = (x^i)_{i \in I} \in \widehat{X}$, for all $p \in \mathbb{R}^{L\overline{S}}$ such that $p(s) \neq 0,\ \forall s \in \overline{S}$, and $\overline{p}_1(x^i - \omega^i) \in \mathrm{Im}\, R$,[3] for every $i \in I$, there is some $y \in \mathbb{R}^{L\overline{S}I}$ such that:

– $\sum_{i \in I} y^i = 0$ and $\overline{p}_1 y^i \in \mathrm{Im}\, R,\ \forall i \in I$,

– $\forall\, i \in I_1,\ \omega^i + y^i \geq 0$,

– $\forall\, i \in I_2,\ (x^i + y^i) \in \mathrm{cl}\, P^i(x)$, and there exists $i_0 \in I_2$ such that $(x^{i_0} + y^{i_0}) \in P^{i_0}(x)$.

Assumption **IR** is the irreducibility assumption introduced by McKenzie [6], [7] and adapted here to the incomplete market framework. What this assumption requires is that whatever subgroup of agents we consider, there always be one subgroup which has goods that they can offer to the complementary subgroup and make every one of these agents better off, with one strictly better off. The incomplete market setting requires for transfers to be achievable on the financial market. What is to notice here is that in a complete market setting, every commodity vector is achievable by the market structure, thus the conditions relying on the commodity

3. Im R denotes the vector space generated by the columns of the matrix R.

price vector p are always satisfied and we get back to the McKenzie irreducibility notion of a pure exchange economy. For a discussion of this irreducibility notion, see McKenzie [8] and Florenzano [3].

In case of existence of a desirable direction $e \in \mathbb{R}^L_+$ (assumption **C2'**), the condition $p(s) \neq 0,\ \forall s \in \overline{S}$ in **IR** can be replaced by $p(s) \cdot e > 0,\ \forall s \in \overline{S}$ and **IR** weakened to:

IR' For every non trivial partition $\{I_1, I_2\}$ of I, for all $x = (x^i)_{i \in I} \in \widehat{X}$, for all $p \in \mathbb{R}^{L\overline{S}}$ such that $p(s) \cdot e > 0,\ \forall s \in \overline{S}$, and $\overline{p}_1(x^i - \omega^i) \in \operatorname{Im} R$, for every $i \in I$, there is some $y \in \mathbb{R}^{L\overline{S}I}$ with the same properties as above in **IR**.

If preferences are strictly monotone at every component of an attainable consumption allocation (assumption **C2"**), the condition $p(s) \neq 0,\ \forall s \in \overline{S}$ in **IR** can be replaced by $p \gg 0$ and **IR** weakened to:

IR" For every non trivial partition $\{I_1, I_2\}$ of I, for all $x = (x^i)_{i \in I} \in \widehat{X}$, for all $p \in \mathbb{R}^{L\overline{S}}$ such that $p \gg 0$, and $\overline{p}_1(x^i - \omega^i) \in \operatorname{Im} R$, for every $i \in I$, there is some $y \in \mathbb{R}^{L\overline{S}I}$ with the same properties as above in **IR**.

Under the previous assumptions, two main results will be proved in this paper.

THEOREM 2.3: *Let ε be a financial economy satisfying the previous assumptions C1, C2 (C2', C2"), F1, F2 and SS1, IR (resp. IR', IR"). Then ε has a financial equilibrium $(\overline{p}, \overline{q}, \overline{x}, \overline{z})$ with $\overline{q} \in \mathbb{Q}$ and $\overline{p}(s) \neq 0,\ \forall s \in \overline{S}$ (resp. $p(s) \cdot e > 0$, $\forall s \in \overline{S}$, $p \gg 0$).*

In a tradition which goes back to Cass [1], we will also prove that any no-arbitrage asset price vector can be embedded in an equilibrium of the economy. More precisely, the theorem to be proved is the following:

THEOREM 2.4: *Let ε be a financial economy satisfying assumptions C1, C2 (C2',C2"), F1, F2 and C3, IR (resp. IR', IR"). Given $q \in \mathbb{Q}$, ε has a financial equilibrium $(\overline{p}, \overline{q}, \overline{x}, \overline{z})$ with $\overline{q} = q$ and $\overline{p}(s) \neq 0,\ \forall s \in \overline{S}$ (resp. $p(s) \cdot e > 0$, $\forall s \in \overline{S}$, $p \gg 0$).*

3 Existence of financial equilibrium

3.1 Preliminary Results

These preliminary results gives, under suitable conditions, a characterization of an equilibrium, and will enable us to introduce the definition of a quasiequilibrium, the weaker notion of equilibrium that we consider.

LEMMA 3.1: *Given* $(\overline{p},\overline{q}) \in \mathbb{R}^{L\overline{S}} \times \mathbb{R}^{J}$, *if for some* $i \in I$, $(\overline{x}^{i},\overline{z}^{i}) \in X^{i} \times Z^{i}$ *satisfies the following condition:*

(3.1) $x^{i} \in P^{i}\left(\overline{x}\right)$ and $\overline{p}_{1}(x^{i}-\omega^{i}) \leq Rz^{i} \Rightarrow \overline{p}(0)\cdot(x^{i}(0)-\omega^{i}(0))+\overline{q}\cdot z^{i} > 0,$

then, $\left(\overline{x}^{i},\overline{z}^{i}\right)$ solves (M^{i}).

Moreover, if $(\overline{p},\overline{q},\overline{x},\overline{z})$ *is an equilibrium, then, for every agent* $i \in I$, *condition 3.1 is satisfied, and, under C1-C2, one has:*

(3.2) $$\overline{p}(\overline{x}^{i}-\omega^{i}) = W(\overline{q})\overline{z}^{i},$$

(3.3) $$\forall s \in \overline{S},\ (\overline{x}^{i}_{-s},x^{i}(s)) \in P^{i}(\overline{x}) \Rightarrow \overline{p}(s)\cdot x^{i}(s) > \overline{p}(s)\cdot \overline{x}^{i}(s).$$

PROOF: We begin by showing that if condition (3.1) is satisfied, then for agent $i \in I$, $(\overline{x}^{i},\overline{z}^{i})$ solves (M^{i}). Suppose that there exists $i \in I$ such that $(\overline{x}^{i},\overline{z}^{i})$ does not solve (M^{i}) at prices $(\overline{p},\overline{q})$, i.e. there exists $(x^{i},z^{i}) \in B^{i}(\overline{p},\overline{q})$ such that $x^{i} \in P^{i}(\overline{x})$: this contradicts condition (3.1).
Assume now that $(\overline{p},\overline{q},\overline{x},\overline{z})$ is an equilibrium. It is immediate that condition (3.1) is satisfied. Since markets clear, using assumptions **C1-C2**, condition (3.2) is satisfied. Finally, let us consider an agent $i \in I$ and an allocation $x'^{i} \in X^{i}$ differing from $\overline{x}^{i}$ only at state *s* such that: $(\overline{x}^{i}_{-s},x^{i}(s)) \in P^{i}(\overline{x})$ and $\overline{p}(s)\cdot x'^{i}(s) \leq \overline{p}(s)\cdot \overline{x}^{i}(s)$. Condition (3.2) implies, $\overline{p}(s)\cdot x'^{i}(s) \leq \overline{p}(s)\cdot \omega^{i}(s)+W_{s}(\overline{q})\cdot \overline{z}^{i}$. By construction of x'^{i}, for the other states $s' \neq s$, $\overline{p}(s')\cdot x'^{i}(s') = \overline{p}(s')\cdot \omega^{i}(s')+W_{s}(\overline{q})\overline{z}^{i}$. Thus, we have $x'^{i} \in P^{i}(\overline{x})$ and $\overline{p}(x'^{i}-\omega^{i}) \leq W(\overline{q})\overline{z}^{i}$: this contradicts the optimality of $(\overline{x}^{i},\overline{z}^{i})$ in $B^{i}(\overline{p},\overline{q})$.

In view of this lemma, let us now define the concept of quasiequilibrium:

DEFINITION 3.1: *The collection* $(\overline{p},\overline{q},\overline{x},\overline{z}) \in \mathbb{R}^{L\overline{S}} \times \mathbb{R}^{J} \times X \times Z$ *is a quasiequilibrium of* ε *if* $\overline{p} \neq 0$, $\sum_{i\in I}\overline{x}^{i} = \sum_{i\in I}\omega^{i}$, $\sum_{i\in I}\overline{z}^{i} = 0$, *and if moreover, for every agent* $i \in I$, *one has:*

(3.1') $x^{i} \in P^{i}(\overline{x})$ *and* $\overline{p}_{1}(x^{i}-\omega^{i}) \geq Rz^{i} \Rightarrow \overline{p}(0)\cdot(x^{i}(0)-\omega^{i}(0))+\overline{q}\cdot z^{i} \geq 0,$

(3.2') $\overline{p}(\overline{x}^{i}-\omega^{i}) = W(\overline{q})\overline{z}^{i},$

(3.3') $\forall s \in \overline{S},\ (\overline{x}^{i}_{-s},x^{i}(s)) \in P^{i}(\overline{x}) \Rightarrow \overline{p}(s)\cdot x^{i}(s) \geq \overline{p}(s)\cdot \overline{x}^{i}(s).$

REMARK 3.1: *The definition of a quasiequilibrium is identical to the definition of equilibrium except that in the preference maximization conditions ((3.1) and (3.3)) the fact that anything preferred to* $\bar{x}^i$ *must cost more is replaced by the weaker requirement that anything preferred to* $\bar{x}^i$ *cannot cost less. It is obvious that any equilibrium is a quasiequilibrium.*

REMARK 3.2: *In view of Assumptions **C1** and **C2**, at a quasiequilibrium* $(\bar{p},\bar{q},\bar{x},\bar{z})$, *we also have the following relation:*

$$x^i \in \text{cl}\, P^i(\bar{x}) \text{ and } \bar{p}_1(x^i - \omega^i) \leq Rz^i \Rightarrow \bar{p}(0)\cdot(x^i(0) - \omega^i(0)) + \bar{q}\cdot z^i \geq 0.$$

The proof of Theorem 2.3 and Theorem 2.4 will proceed in two main steps. We will begin by proving, under the standard assumptions, the existence of a quasi-equilibrium. Then, under specific survival and irreducibility assumptions, we will prove that any such quasiequilibrium is achieved as an equilibrium.

3.2 Existence of quasiequilibrium

For the proof of Theorem 2.3, we take as departure point the following result:

PROPOSITION 3.2: *If we assume that for all* $i \in I$, $\omega^i \gg 0$, *then, under assumptions C1-C2, F1-F2, there exists a financial equilibrium* $(\bar{p},\bar{q},\bar{x},\bar{z})$ *with* $\bar{q} \in \mathbb{Q}$, $\|\bar{p}(s)\| = 1$, $\forall s \in S$, *and* $\bar{p}(0) \neq 0$.

This result is easily proved from Proposition 4.2 in Florenzano [2] which gives conditions for existence of equilibrium in models with bounded portfolios.[4]

As well known, the same result can also be viewed as a consequence of the existence of equilibrium for an economy with numeraire assets whose *R* is the return matrix denominated in units of a numeraire. In this approach, besides **C1-C2** and **F1-F2**, it should also be assumed that the numeraire is desirable at each state of nature and each component of an attainable allocation. Such an assumption is for example satisfied when we deal with strictly monotone preferences.

PROPOSITION 3.3: *Under assumptions **C1-C2, F1-F2,** the financial economy* ε *has a quasiequilibrium* $(\bar{p},\bar{q},\bar{x},\bar{z})$ *such that* $\bar{q} \in \text{cl}\,\mathbb{Q}$, $\|\bar{p}(0)\| + \|\bar{q}\| = 1$ *and* $\|\bar{p}(s)\| = 1$, *for every state* $s \in S$.

4. Considering the following modified economies $\varepsilon^\nu = (X^i, P^i, \omega^i, Z^{i\nu}; R)$ where $Z^{i\nu} = \{z \in \mathbb{R}^J \mid z_j \geq -\nu, \forall j \in J\}$, following Proposition 4.2 in Florenzano [2], each ε^ν has an equilibrium $(\bar{p}^\nu, \bar{q}^\nu, \bar{x}^\nu, \bar{z}^\nu)$ such that $\|\bar{p}^\nu(s)\| = 1,\ \forall s \in S,\ \bar{p}^\nu(0) \neq 0$. After normalization of commodity and asset prices at period 0 by $\|\bar{p}^\nu(0)\| + \|\bar{q}^\nu\| = 1$, , it is easily seen that the sequence $(\bar{p}^\nu, \bar{q}^\nu, \bar{x}^\nu, \bar{z}^\nu)$ has a subsequence converging to $(\bar{p},\bar{q},\bar{x},\bar{z})$ and that $(\bar{p},\bar{q},\bar{x},\bar{z})$ is a financial equilibrium of ε.

Proof. Let $(\omega^{in})_{i\in I}$ be a sequence of initial endowments satisfying $\omega^{in} \gg 0$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} \omega^{in} = \omega^i$. For each $n \in \mathbb{N}$, we consider the financial economy:

$$\varepsilon^n = \left((X^i, P^i, \omega^{in}, Z^i)_{i\in I}, R\right).$$

Recalling Proposition 3.2, there exists an equilibrium $\left(\overline{p}^n, \overline{q}^n, \overline{x}^n, \overline{z}^n\right)$ which satisfies $\left\|\overline{p}^n(s)\right\| = 1,\ \forall_s \in S$ and $\left\|\overline{p}^n(0)\right\| + \left\|\overline{q}^n\right\| = 1$. It follows that for every $s \in S$, there is some subsequence of $\overline{p}^n(s)$ which converges to $\overline{p}(s)$, with $\left\|\overline{p}(s)\right\| = 1$. As well, there is some subsequence of $(\overline{p}^n(0), \overline{q}^n)$ which converges to $(\overline{p}(0), \overline{q})$, with $\left\|\overline{p}(0)\right\| + \left\|\overline{q}\right\| = 1$ and $q \in \mathrm{cl}\,\mathbb{Q}$. Let $\varepsilon > 0$. From $\lim_{n\to\infty} \omega^{in} = \omega^i$, we deduce that for all *n* large enough, we have $0 \leq \sum_{i\in I} \overline{x}^{in} \leq \sum_{i\in I} \omega^i + \varepsilon I$.[5] Thus, for each $i \in I$,, there is a subsequence of $\overline{x}^{in}$ which converges to some $\overline{x}^i \geq 0$. Without loss of generality, we can assume that $(\overline{p}^n, \overline{q}^n, \overline{x}^n)$ converges to $(\overline{p}, \overline{q}, \overline{x})$. Since $(\overline{p}^n, \overline{q}^n, \overline{x}^n, \overline{z}^n)$ is an equilibrium of ε^n, it satisfies $\overline{p}_1^n(\overline{x}^{in} - \omega^{in}) = R\overline{z}^{in}$. Recalling that Assumption **F1** states rank $R = J$, the matrix *R* has a left inverse. Thus, for every agent $i \in I$, $\overline{z}^{in}$ converges to some $\overline{z}^i$. Moreover, $\sum_{i\in I} \overline{z}^i = 0$. Finally, we have: For all $i \in I$, $\lim_{n\to\infty} \overline{x}^{in} = \overline{x}^i$, $\lim_{n\to\infty} \overline{z}^{in} = \overline{z}^i$, and $\lim_{n\to\infty} \overline{p}^n = \overline{p}$, with $\left\|\overline{p}(s)\right\| = 1,\ \forall_s \in S$ and $\lim_{n\to\infty} \overline{q}^n = \overline{q}$ with $\left\|\overline{p}(0)\right\| + \left\|\overline{q}\right\| = 1$. Moreover, $\sum_{i\in I} \overline{x}^i = \sum_{i\in I} \omega^i$, $\sum_{i\in I} \overline{z}^i = 0$.

Passing to the limits in the relation $\overline{\overline{p}}^n(\overline{x}^{in} - \omega^{in}) = W(\overline{q}^n)\overline{z}^{in}$, for each agent $i \in I$, we get: $\overline{\overline{p}}(\overline{x}^i - \omega^i) = W(\overline{q})\overline{z}^i$.

Assume now that there exists a bundle $(x^i, z^i) \in X^i \times Z^i$ satisfying $x^i \in P^i(\overline{x})$ and $\overline{\overline{p}}_1(x^i - \omega^i) \leq Rz^i$. In view of assumption **F1**, letting $\lambda > 0$, we have: $\overline{\overline{p}}_1(x^i - \omega^i) \ll R(z^i + \lambda\hat{z})$, thus we deduce that for *n* big enough, $\overline{\overline{p}}_1^n(x^i - \omega^{in}) \ll R(z^i + \lambda\hat{z})$. From the fact that $(\overline{x}^{in}, \overline{z}^{in})$ is optimal in $B^i(\overline{p}^n, \overline{q}^n)$ it follows from Lemma 3.1 that $\overline{p}^n(0)\cdot(x^i(0) - \omega^{in}(0)) + q^n\cdot(z^i + \lambda\hat{z}) > 0$. Letting *n* tend to infinity, one gets $\overline{p}(0)\cdot(x^i(0) - \omega^i(0)) + q\cdot(z^i + \lambda\hat{z}) \geq 0$. By letting λ tend to 0, we also get: $\overline{p}(0)\cdot(x^i(0) - \omega^i(0)) + q\cdot z^i \geq 0$.

Finally, let $s \in \overline{S}$ and let us consider $x^i \in X^i$ differing from $\overline{x}^i$ only at state *s* and such that $(x^i(s), \overline{x}^i_{-s}) \in P^i(\overline{x})$. For *n* big enough, $(x^i(s), \overline{x}^{in}_{-s}) \in P^i(\overline{x}^n)$ and, in view of Lemma 3.1, we have $\overline{p}^n(s)\cdot x^i(s) > \overline{p}^n(s)\cdot \overline{x}^{in}(s)$. Passing to limits, we get: $\overline{p}(s)\cdot x^i(s) \geq \overline{p}(s)\cdot \overline{x}^i(s)$.

---

5. The vector *I* denotes a vector of $\mathbb{R}^{L\bar{S}}$ having each component equal to one.

The following proposition proves that any no-arbitrage asset price vector can be embedded in a quasiequilibrium.

PROPOSITION 3.4: *Under the assumptions **C1-C2, F1-F2**, given* $\bar{q} = \sum_{s\in \bar{S}} \lambda_s R(s)$ *and* $i_0 \in I$, *the financial economy* ε *has a quasiequilibrium* $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ *such that* $\bar{q} = q$, $\bar{p} \neq 0$, *and* $P^{i_0}(\bar{x}) \cap \left\{ x^{i_0} \in X^i \mid \sum_{s\in \bar{S}} \lambda_s \bar{p}(s)\cdot(x^{i_0}(s) - \omega^{i_0}(s)) < 0 \right\} = \emptyset$.

Proof. Let $\bar{q}$ be the no-arbitrage asset price vector $\bar{q} = \sum_{s\in S} \lambda_s R(s)$, and let $(\omega^{in})_{i\in I}$ be a sequence of initial endowments satisfying, for each $n \in \mathbb{N}$, $\omega^{in} \gg 0$ and $\lim_{n\to\infty} \omega^{in} = \omega^i$. Let $i_0 \in I$. From Theorem 7.1 of Florenzano [2], there exists an equilibrium $\left(\bar{p}^n, \bar{q}, \bar{x}^n, \bar{z}^n\right)$ of the modified financial economy ε^n.

$$\varepsilon^n = \left((X^i, P^i, \omega^{in}, Z^i)_{i\in I}, R\right),$$

such that $\bar{x}^{i_0}$ is also optimal in his Arrow-Debreu budget set:

$$B^{i_0}(p,\lambda) = \left\{ x^{i_0} \in X^i \mid \sum_{s\in \bar{S}} \lambda_s p(s)\cdot(x^i(s) - \omega^i(s)) \leq 0 \right\}.$$

For each $n \in \mathbb{N}$, the price vector $\bar{p}^n$ is such that $\left\|\bar{\pi}^n\right\| = 1$, where $\bar{\pi}^n = \left(\lambda_s \bar{p}^n(s)\right)_{s\in\bar{S}}$. Thus, there is some subsequence of $\bar{p}^n$ which converges to $\bar{p}$, with $\bar{p} \neq 0$. From $\lim_{n\to\infty} \omega^{in} = \omega^i$, exactly as in the proof of Proposition 3.3, we deduce that for each $i \in I$, some subsequence of $\bar{x}^{in}$ converges to some $\bar{x}^i \geq 0$. Without loss of generality, we can assume that $(\bar{p}^n, \bar{x}^n)$ converges to $(\bar{p}, \bar{x})$. Since $(\bar{p}^n, \bar{q}, \bar{x}^n, \bar{z}^n)$ is an equilibrium of ε^n, it satisfies $\overline{p_1^n}(\bar{x}^{in} - \omega^{in}) = R\bar{z}^{in}$. From Assumption **F1**, and analogous to the proof of Proposition 3.3, for every agent $i \in I$, $\bar{z}^{in}$ converges to some $\bar{z}^i$. Finally, we have: For all $i \in I$, $\lim_{n\to\infty} \bar{x}^{in} = \bar{x}^i$, $\lim_{n\to\infty} \bar{z}^{in} = \bar{z}^i$, and $\lim_{n\to\infty} \bar{p}^n = \bar{p}$, with $\bar{p} \neq 0$. Moreover, $\sum_{i\in I} \bar{x}^i = \sum_{i\in I} \omega^i$, $\sum_{i\in I} \bar{z}^i = 0$, and for each $i \in I$, passing to the limits in the relation $\overline{\bar{p}^n}(\bar{x}^{in} - \omega^{in}) = W(\bar{q})\bar{z}^{in}$, we get: $\bar{\bar{p}}(\bar{x}^i - \omega^i) = W(\bar{q})\bar{z}^i$.

Assume now that there exists a bundle $(x^i, z^i) \in X^i \times Z^i$ satisfying $x^i \in P^i(\bar{x})$ and $\overline{\bar{p}_1}(x^i - \omega^i) \leq Rz^i$. As in the proof of Proposition 3.3, one proves that $\bar{p}(0)\cdot(x^i(0) - \omega^i(0)) + q\cdot z^i \geq 0$. Moreover, if $P^{i_0}(\bar{x}) \cap \{x^{i_0} \in$

$X^i \mid \sum_{s\in\overline{S}} \lambda_s p(s)\cdot(x^{i_0}(s)-\omega^{i_0}(s))<0\} \neq \emptyset$, then for n large enough, there exists $x^{i_0} \in P^{i_0}(\overline{x}^n)$ and $\sum_{s\in\overline{S}} \lambda_s p^n(s)\cdot\left(x^{i_0}(s)-\omega^{i_0 n}(s)\right)<0$ which contradicts the optimality of the equilibrium consumption $\overline{x}^{i_0 n}$ in its Arrow-Debreu budget set.

Finally, let $s\in\overline{S}$ and let us consider $x^i \in X^i$ differing from $\overline{x}^i$ only at state s such that $(x^i(s),\overline{x}^i_{-s})\in P^i(\overline{x})$. For n big enough, $(x^i(s),\overline{x}^{in}_{-s})\in P^i(\overline{x}^n)$ and recalling the result of Lemma 3.1, we have $\overline{p}^n(s)\cdot x^i(s) > \overline{p}^n(s)\cdot \overline{x}^{in}(s)$. Passing to limits, we get: $\overline{p}(s)\cdot x^i(s)\geq \overline{p}(s)\cdot\overline{x}^i(s)$.

3.3 **From quasiequilibrium to equilibrium**

This section will highlight how the specific assumptions **SS1** (**C3**) and **IR** interfere in order to achieve the quasiequilibrium $(\overline{p},\overline{q},\overline{x},\overline{z})$ obtained in Proposition 3.3 (resp. Proposition 3.4) as an equilibrium.

THEOREM 3.5: *Under **C1-C2**, **SS1 (C3)** and **IR**, every quasiequilibrium $(\overline{p},\overline{q},\overline{x},\overline{z})$ of ε obtained in Proposition 3.3 (respectively Proposition 3.4), is an equilibrium of ε. If **C2** is replaced by **C2'** (resp. **C2"**), **IR** can be replaced by **IR"** (resp. **IR"**).*

In both cases, let $(\overline{p},\overline{q},\overline{x},\overline{z})$ be a quasiequilibrium and let $\{I_1,I_2\}$ be such that:

$$I_1=\left\{i\in I:\left\{(x^i,z^i)\in X^i\times Z^i \left| \begin{array}{l}\overline{p}(0)\cdot(x^i(0)-\omega^i(0))+\overline{q}\cdot z^i<0\\ \overline{p}_1(x^i-\omega^i)-Rz^i\leq 0\end{array}\right. \right\}=\emptyset\right\}.$$

$$I_2=\left\{i\in I:\left\{(x^i,z^i)\in X^i\times Z^i \left| \begin{array}{l}\overline{p}(0)\cdot(x^i(0)-\omega^i(0))+\overline{q}\cdot z^i<0\\ \overline{p}_1(x^i-\omega^i)-Rz^i\leq 0\end{array}\right. \right\}\neq\emptyset\right\}.$$

LEMMA 3.6: *Under **C1-C2** and (**SS1** or **C3**), $I_2\neq\emptyset$. Moreover, for every agent i belonging to the set I_2, $(\overline{x}^i,\overline{z}^i)$ is optimal in $B^i(\overline{p},\overline{q})$. Therefore, for all state $s\in\overline{S}$, $\overline{p}(s)\neq 0$. Analogously, if **C2** is replaced by **C2'** (**C2"**) the commodity price vector satisfies $\overline{p}(s)\cdot e>0$, for all state $s\in\overline{S}$ (resp. $\overline{p}\gg 0$).*

Proof. If $(\overline{p},\overline{q},\overline{x},\overline{z})$ is a quasiequilibrium obtained by Proposition 3.3, it follows from **SS1** that $I_2\neq\emptyset$.

On the other hand, according to **C3**, for some $i_0 \in I$, $\omega^{i_0} \gg 0$. Let us consider $(\overline{p}, \overline{q}, \overline{x}, \overline{z})$ a quasiequilibrium obtained by Proposition 3.4 such that:

$$P^{i_0}(\overline{x}) \cap \left\{ x^{i_0} \in X^i \mid \sum_{s \in \overline{S}} \lambda_s p(s) \cdot (x^{i_0}(s) - \omega^{i_0}(s)) < 0 \right\} = \emptyset.$$

Since $\overline{p} \neq 0$, the set $\left\{ x^{i_0} \mid \sum_{s \in \overline{S}} \lambda_s \overline{p}(s) \cdot (x^{i_0}(s) - \omega^{i_0}(s)) < 0 \right\}$ is a non empty set. Thus, $\overline{x}^{i_0}$ is optimal in his Arrow-Debreu budget set, and in view of Assumption **C2**, $\overline{p}(s) \neq 0$, $\forall s \in \overline{S}$. Using again **C3** and taking for each $s \in \overline{S}$, $x^{i_0}(s)$ such that $\overline{p}(s) \cdot (x^{i_0}(s) - \omega^{i_0}(s)) < 0$, one sees that $I_2 \neq \emptyset$.

In both cases, let $i \in I_2$. We now show that the bundle $(\overline{x}^i, \overline{z}^i)$ is optimal for agent i in his budget set $B^i(\overline{p}, \overline{q})$. Indeed, assume that there exists $(x^i, z^i) \in X^i \times Z^i$ such that $x^i \in P^i(\overline{x})$ and $(x^i, z^i) \in B^i(\overline{p}, \overline{q})$. Since $i \in I_2$, there exists a bundle $(\tilde{x}^i, \tilde{z}^i) \in X^i \times Z^i$ such that $\overline{p}(0) \cdot (\tilde{x}^i(0) - \omega^i(0)) + \overline{q} \cdot \tilde{z}^i < 0$ and $\overline{p}_1(\tilde{x}^i - \omega^i) + R\tilde{z}^i \leq 0$. Let $\lambda \in [0, 1[$ and consider the following convex combination:

$$\begin{cases} x^{i\lambda} = \lambda x^i + (1-\lambda)\tilde{x}^i \\ z^{i\lambda} = \lambda z^i + (1-\lambda)\tilde{z}^i, \end{cases}$$

For λ close enough to 1, and in view of Assumption **C1**, $x^{i\lambda} \in P^i(\overline{x})$. Moreover, one has $\overline{p}_1(x^{i\lambda} - \omega^i) \leq Rz^{i\lambda}$ and $\overline{p}(0) \cdot (x^{i\lambda}(0) - \omega^i(0)) + \overline{q} \cdot z^{i\lambda} < 0$, which contradicts the fact that $(\overline{p}, \overline{q}, \overline{x}, \overline{z})$ is a quasiequilibrium.

The last assertion (i.e. $\overline{p}(s) \neq 0$, $\forall s \in \overline{S}$) follows from Assumption **C2** and the previous assertions.

LEMMA 3.7: *Under **IR** (resp. **IR'**, **IR"**),* $I_1 = \emptyset$.

Proof. Assume that $I_1 \neq \emptyset$. The partition $\{I_1, I_2\}$ is non trivial and from Assumption **IR**, there exists $y \in \mathbb{R}^{L\overline{S}I}$ and $z^i \in \mathbb{R}^J$ such that: $\sum_{i \in I} y^i = 0$ and $\overline{p}_1 y^i = Rz^i$, $\forall_i \in I$. Furthermore, $\forall i \in I_1$, $y^i + \omega^i \geq 0$, and $\forall i \in I_2$, $(\overline{x}^i + y^i) \in \text{cl}\, P^i(\overline{x})$, and $\exists i_0 \in I_2$, $(\overline{x}^{i_0} + y^{i_0}) \in P^{i_0}(\overline{x}^{i_0})$. Let an agent $i \in I_2$. Since $(\overline{p}, \overline{q}, \overline{x}, \overline{z})$ is a quasi-equilibrium, knowing that $(\overline{x}^i + y^i) \in \text{cl}\, P^i(\overline{x})$, and $\overline{p}_1(\overline{x}^i + y^i - \omega^i) = R(\overline{z}^i + z^i)$, Remark 3.2 implies $\overline{p}(0) \cdot (\overline{x}^i(0) + y^i(0) - \omega^i(0)) + \overline{q} \cdot (\overline{z}^i + z^i) \geq 0.$. Moreover, from the previous lemma, agents of I_2 are optimal in their budget sets, hence $(\overline{x}^{i_0} + y^{i_0}) \in P^{i_0}(\overline{x})$ and $\overline{p}_1(\overline{x}^{i_0} + y^{i_0} - \omega^{i_0}) = R(\overline{z}^{i_0} + z^{i_0})$ implies

$\bar{p}(0)\cdot(\bar{x}^{i_0}(0)+y^{i_0}(0)-\omega^{i_0}(0))+\bar{q}\cdot(\bar{z}^{i_0}+z^{i_0})>0$. Summing among all agents of I_2, one has: $\bar{p}(0)\cdot\sum_{i\in I_2}\left(\bar{x}^i(0)+y^i(0)-\omega^i(0)\right)+\quad \bar{q}\cdot\sum_{i\in I_2}\left(\bar{z}^i+z'^i\right)>0.$ Since $\sum_{i\in I}\bar{x}^i=\sum_{i\in I}\omega^i$, $\sum_{i\in I}y^i(0)=0$, $\sum_{i\in I}z^i=\sum_{i\in I}\bar{z}^i=0$, we deduce that $\bar{p}(0)\cdot\sum_{i\in I_1}\left(\bar{x}^i(0)+y^i(0)-\omega^i(0)\right)+\bar{q}\cdot\sum_{i\in I_1}\left(\bar{z}^i+z'^i\right)<0$. Consequently, there exists $i_1\in I_1$ such that $\bar{p}(0)\cdot(\bar{x}^{i_1}(0)+y^{i_1}(0)-\omega^{i_1}(0))+\bar{q}\cdot(\bar{z}^{i_1}+z'^{i_1})<0$. This last inequality can be written as: $\bar{p}(0)\cdot(\bar{x}^{i_1}(0)-\omega^{i_1}(0)+y^{i_1}(0)+\omega^{i_1}(0)-\omega^{i_1}(0))+\bar{q}\cdot(\bar{z}^{i_1}+z'^{i_1})<0$. Since $\bar{p}(0)\cdot(\bar{x}^{i_1}(0)-\omega^{i_1}(0))=\bar{q}\cdot\bar{z}^{i_1}$, we deduce: $\bar{p}(0)\cdot(y^{i_1}(0)+\omega^{i_1}(0)-\omega^{i_1}(0))+\bar{q}\cdot z^{i_1}<0$. Recalling that $\overline{\overline{p_1}}(y^{i_1}+\omega^{i_1}-\omega^{i_1})=Rz^{i_1}$, we get a contradiction with the definition of I_1. Hence, $I_1=\emptyset$.

The proof of Theorems 2.3 and 2.4 follows immediately from these two lemmas.

4. Discussion of Survival and Irreducibility Assumptions

Our main focus in this section is to see how these specific survival and irreducibility assumptions are related to the fundamentals of the model.

4.1 Survival Assumption

Let us consider the following conditions:

C3’ $\left(\sum_{i\in I}\omega^i(0)\gg 0\right)$ and ($\exists i\in I,\ \omega^i(s)\gg 0,\ \forall s\in S$).

Remark here that **C3** implies **C3’**.

LEMMA: *Under **F1**, **C3’**$\Rightarrow$ **SS1**.*

Proof. Let $(p,q)\in\mathbb{R}^{L\bar{S}}\times\mathbb{R}^J$, satisfying $p(s)\neq 0\quad\forall s\in S,\quad q\in \text{cl}\,\mathbb{Q}$, and $(p(0),q)\neq(0,0)$. Two cases may occur: either $p(0)=0$ or $p(0)\neq 0$.

If $p(0)=0$, then we know that $q\neq 0$. From **C3’**, there exists an agent $i\in I$ satisfying $\omega^i(s)\in \text{int}\,X^i(s)$, $\forall s\in S$. From this, we easily deduce that there exists

$x^i \in X^i$, $\overline{p}_1(x^i - \omega^i) \ll 0$. Moreover, from **F1**, there exists $\hat{z} \in \mathbb{R}^J$ such that $R\hat{z} \gg 0$. Let $\mu > 0$ be such that:

$$\overline{p}_1(x^i - \omega^i) \ll -R(\mu\hat{z}) \ll 0.$$

Since $q \in \mathrm{cl}\,\mathbb{Q} \setminus \{0\}$, there exists $\lambda \in \mathbb{R}^S_+$, $\lambda \neq 0$ such that $q = \sum_{s\in S} \lambda_s R(s)$. Hence, $q \cdot (-\mu\hat{z}) = \sum_{s\in S} \lambda_s R(s)(-\mu\hat{z}) < 0$, and **SS1** holds.

If $p(0) \neq 0$, since $\sum_{i\in I} \omega^i(0) \gg 0$, i.e. $\sum_{i\in I} \omega^i(0) \in \mathrm{int} \sum_{i\in I} X^i(0)$. Thus, for every agent $i \in I$, there exists a first period consumption vector $x^i(0) \in X^i(0)$ such that $p(0)\cdot\left(\sum_{i\in I} x^i(0) - \sum_{i\in I} \omega^i(0)\right) < 0$. Hence, for an agent $i \in I$, $p(0)\cdot(x^i(0) - \omega^i(0)) < 0$. By choosing a portfolio $z^i = 0$, and a second period consumption vector such that $x^i(s) = \omega^i(s)$, $\forall s \in S$, **SS1** holds.

4.2 Irreducibility assumption

Let us remind here that, as pointed out in section 2.3, the irreducibility assumption changes considering what is assumed on preferences.

4.2.1 Strong Monotonicity

LEMMA 4.2. *Assume that **C2"**, **F1** and **SS1** are guaranteed. Then we get the following implication:*

$$\forall i \in I,\ \left(\omega^i(0) > 0\right) \text{ or } \left(\omega^i(s) > 0, \forall s \in S\right) \Rightarrow \textbf{IR}''.$$

Proof. Let $\{I_1, I_2\}$ be a non-trivial partition of set I. Let $p(s) \gg 0$, $\forall s \in S$. Assume first that in subset I_1, there exists an agent i_0 such that $\omega^{i_0}(0) > 0$, we set $y^i(0) = -\omega^i(0)$, and give $-y^i(0)$ to an agent of I_2. We get that:

$$\begin{cases} \omega^i(0) + y^i(0) \geq 0, & \forall i \in I_1 \\ \left(x^i - y^i\right) \in P^i(x), & \forall i \in I_2 \end{cases}$$

By setting $y^i(s) = 0$, $\forall s \in S$, , **IR"** holds.

If there is no agent in I_1 such that $\omega^{i_0}(0) > 0$, then, since $\forall i \in I,\ \left(\omega^i(0) > 0\right)$ or ($\omega^i(s) > 0,\ \ \forall s \in S$), we get that $\omega^i(s) > 0,\ \ \forall s \in S,\ \ \forall i \in I_1$. From **F1**, there exists $\hat{z} \in \mathbb{R}^J$ such that $R\hat{z} \gg 0,$ we may then choose a scalar $\lambda > 0$ such that:

$$0 \ll R\lambda\hat{z} \ll \overline{p}_1\omega^{i_0}.$$

For all $s \in S$, we can choose $y^{i_0}(s) \in \mathbb{R}^L_-$ such that:

$$\begin{cases} p(s)\cdot y^{i_0}(s) = R(s)\left(-\lambda\hat{z}\right) \\ \omega^{i_0}(s) + y^{i_0}(s) \geq 0 \end{cases}$$

Let us give $-y^{i_0}(s)$ to an agent *i* of I_2. Recalling **C2"**, and by setting $y^i(0) = 0$, we get $(x^i - y^{i_0}) \in P^i(x)$, and **IR"** holds.

As noticed by Gottardi and Hens, if we join to **C3'** the assumption that:

$$\forall i \in I,\ \left(\omega^i(0) > 0\right) \text{ or } \left(\omega^i(s) > 0, \forall s \in S\right),$$

we get a set of assumptions weaker than the ones considered by Werner [10].

4.2.2. Existence of a Desirable Direction

We now consider that preferences of agents satisfy assumption **C1** and **C2'**. We get the following result:

LEMMA 4.3. *Under **C1, C2', F1** and **C3'**, we get the following implication:*

$$\forall i \in I,\ \left(\omega^i(0) \gg 0\right) \text{ or } \left(\omega^i(s) \gg 0, \forall s \in S\right) \Rightarrow IR'.$$

Proof. Let $\{I_1, I_2\}$ be a non-trivial partition of the set of agents *I*. Let $e \in \mathbb{R}^L$ given by assumption **C2'**. Let $p \in \mathbb{R}^{L\overline{S}}$ such that $p(s)\cdot e > 0,\ \ \forall s \in S$ and $x \in \widehat{X}$ be an attainable allocation on the financial market in second period by all agents, i.e. $\forall i \in I$, there exists a portfolio $z^i \in \mathbb{R}^J$ such that $\overline{p}_1(x^i - \omega^i) = Rz^i$.

If in I_1 there exists $i \in I$ such that $\omega^i(0) \gg 0$, from **C2'**, for every agent $i \in I$, there exists a consumption vector $x'^i \in X^i$ such that $x'^i(s') = x^i(s),$ $\forall s' \neq 0,\ \ x'^i(0) = x^i(0) + \lambda e,\ \ \lambda > 0$ and $x'^i \in P^i(x)$. Moreover, since $\forall i \in I_1$, $\omega^i(0) \in \text{int}\, X^i(0)$, we get $\left(\omega^i(0) - \lambda e\right) \in X^i(0)$ if λ small enough. Thus, by choosing $y \in \mathbb{R}^{L\overline{S}I}$ such that:

$y^i(0) = -\lambda e$, $\forall i \in I_1$, $y^i(0) = \lambda e$, $\forall i \in I_2$, and $y^i(s) = 0$, $\forall s \in S$, $\forall i \in I$, **IR'** holds.

If there is no agent in I_1 such that $\omega^i(0) \gg 0$ then, since $\forall i \in I$, $\left(\omega^i(0) \gg 0\right)$ or ($\omega^i(s) \gg 0,\ \forall s \in S$), we get that $\omega^i(s) \gg 0,\ \forall s \in S,\ \forall i \in I_1$. Let $s \in S$. From **C2'**, for every agent $i \in I$, there exists $\lambda > 0$ such that $(x^i(s)+\lambda e, x^i_{-s}) \in P^i(x)$. We do this for every state $s \in S$. Let $\overline{\lambda} > 0$ such that $\overline{\lambda} < \lambda$ and $\forall s \in S$, $\omega^i(s) \gg \overline{\lambda} e$. Since $\overline{p_1} e \gg 0$, we have $\overline{\lambda}\,\overline{p_1} e \gg 0$. Similarly to the previous case, from **F1**, we can find $\hat{z} \in \mathbb{R}^J$ and $\nu \in \mathbb{R}$ such that $\overline{\lambda}\,\overline{p_1} e \ll R(\nu\hat{z})$. We can set $(\gamma_s)_{s\in S} \in]0,1[$ such that $(\gamma_s \overline{\lambda}) p(s) \cdot e = R(s)(\nu\hat{z}),\ \forall s \in S$. In view of **C1**, $(x^i(0), x^i(1)+\gamma_1\overline{\lambda}e, x^i(2)+\gamma_2\overline{\lambda}e, \ldots, x^i(S)+\gamma_S\overline{\lambda}e) \in P^i(x)$. Let $y^i(s) = \gamma_s \overline{\lambda} e$, $\forall s \in S$, $\forall i \in I_2$ and $y^i(s) = -\gamma_s \overline{\lambda} e$, $\forall s \in S$, $\forall i \in I_1$. Noticing that $\omega^i(s) - \gamma_S \overline{\lambda} e \geq 0$, $\forall s \in S$, $\forall i \in I_1$, **IR'** holds. ■

References

Cass, D. (1984): "Competitive Equilibria in Incomplete Financial Markets", CARESS Working Paper No 84-09, University of Pennsylvania.

Florenzano, M. (1999): "General Equilibrium of Financial Markets: An Introduction"; *Cahiers Bleus du CERMSEM, 1999.76*.

Florenzano, M. (2003): "General Equilibrium Analysis - Existence and Optimality Properties of Equilibria"; *Kluwer Acadamic Publishers*.

Gottardi, P. and Hens, T. (1996): "The Survival Assumption and Existence of Competitive Equilibria when Asset Markets are Incomplete", *Journal of Economic Theory*, 313-323.

Gottardi, P. and Hens, T. (1994): "The Survival Assumption and Existence of Competitive Equilibria when Asset Markets are Incomplete", *SFB Discussion Paper A-450, University of Bonn*.

McKenzie, L. (1959): "On the Existence of General Equilibrium for a Competitive Market", *Econometrica, 27, 54-71*.

McKenzie, L. (1961): "On the Existence of General Equilibrium: some corrections", *Econometrica, 29, 247-248*.

McKenzie, L. (1987): "General Equilibrium", The New Palgrave: A Dictionary of Economics, edited by J.Eatwell, M.Milgate and P.Newman, Macmillan, London.

Seghir A. and al. (2001): "The survival assumption and the existence of equilibrium with incomplete markets"; *Cahiers Bleus du CERMSEM, 2001.47*.

Werner, J. (1985): "Equilibrium in Economies with Incomplete Financial Markets", *Journal of Economic Theory* 36, 110-119.

5. Appendix

Taking into account that, in a proof *à la Cass*, we get to a quasiequilibrium price consumption vector $p \neq 0$, if we introduce a condition similar to **SS1**, it would be:

SS2 For all $p \in \mathbb{R}^{L\overline{S}}$, $p \neq 0$, $\forall q \in \mathbb{Q}$, there is some $i \in I$, some $(x^i, z^i) \in X^i \times Z^i$ such that $p(0)\cdot(x^i(0) - \omega^i(0)) + q \cdot z^i < 0$ and $\overline{p}_1(x^i - \omega^i) \geq Rz^i$.

Here is an interesting result:

LEMMA 5.1. ***SS2** implies that markets are complete.*

Proof. We are going to construct a particular portfolio family $z^\sigma \in \mathbb{R}^J$, $\forall \sigma \in S$. Let $\sigma \in S$ and $p \in \mathbb{R}^{L\overline{S}}$ be such that $p(\sigma) \neq 0$ and $p(s) = 0$, $\forall s \in \overline{S} \setminus \{\sigma\}$. From **SS2**, there exists an agent $i \in I$ and a portfolio $z^\sigma \in \mathbb{R}^J$ such that $-q \cdot z^\sigma > 0$ and $R(s)z^\sigma \geq 0$, $\forall s \in S \setminus \{\sigma\}$. By the no-arbitrage condition, we get $R(s)z^\sigma < 0$. Let $t^\sigma = (t_s^\sigma) \in \mathbb{R}^S$ be defined by $t_s^\sigma = \lambda_s R(s) z^\sigma$, $\forall s \in S$. For each $\sigma \in S$, we consider the vector $t^\sigma = \left({}^t\lambda R z^\sigma\right)$. If we sum among all states $s \in S$, we get: $\sum_{s\in S} t_s^\sigma = \sum_{s \in S} \lambda_s R(s) z^\sigma = q \cdot z^\sigma < 0$. Moreover,

$$t^\sigma = \left(\lambda_1 R(1) z^\sigma, \lambda_2 R(2) z^\sigma, \ldots, \lambda_s R(s) z^\sigma, \ldots, \lambda_S R(S) z^\sigma\right).$$

Let us now consider the matrix $M \in \mathcal{M}_{(S,S)}$:

$$M = \begin{pmatrix} t^1 \\ . \\ . \\ . \\ t^S \end{pmatrix} = \begin{pmatrix} \lambda_1 R(1) z^1 & \lambda_2 R(2) z^1 & . & . & . & \lambda_{S-1} R(S-1) z^1 & \lambda_S R(S) z^1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & \lambda_s R(s) z^s & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \lambda_1 R(1) z^S & . & . & . & . & \lambda_{S-1} R(S-1) z^S & \lambda_S R(S) z^S \end{pmatrix}$$

If we denote the components of $M = \left(m_{\sigma s}\right)$, we remark that $\sum_{s \in S, s \neq \sigma} m_{\sigma s} < 0$ and $|m_{\sigma\sigma}| > \sum_{s \in S, s \neq \sigma} m_{\sigma s}$. Thus, M is *a diagonal dominant matrix*, which implies that $\left(Rz^1, \ldots, Rz^S\right)$ is an independent family. Hence, Dim $\operatorname{Im} R = S = J$: markets are complete.