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Overlapping Generations Model with Incomplete Markets: The Numeraire Case II

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ABSTRACT. – This paper deals with the existence of equilibrium in a non-classical Overlapping Generations Model with incomplete markets and one-period numeraire assets. The stochastic structure of the model is given by an infinite event-tree in which each node has a finite number of immediate successors. The peculiarity of this Overlapping Generations Model consists of the eventual presence of infinitely lived agents. To rule out Ponzi schemes, we impose debt constraints, due to LEVINE-ZAME [1996], to the infinitely lived agents.

Modèle à générations imbriquées avec marchés incomplets : Le cas numéraire II

RÉSUMÉ. – Ce papier prouve l’existence de l’équilibre dans un modèle à générations non classique avec des marchés incomplets et des actifs numéraires de maturité une période. La structure stochastique du modèle est caractérisée par un arbre infini d’évènements dans lequel chaque nœud a un nombre fini de successeurs immédiats. La particularité de ce modèle à générations consiste en la présence éventuelle d’agents à durée de vie infinie. Afin d’exclure les mécanismes de cavalerie, nous imposons aux agents à durée de vie infinie des contraintes de dettes, introduites par LEVINE-ZAME [1996].

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1 Introduction

There are two classical manners to extend an economy to an infinite horizon. The first one is to assume that each agent is alive for a finite number of periods and is succeeded by his children forming an Overlapping Generations model. The Overlapping Generations model was introduced by ALLAIS [2] and SAMUELSON [22]. BALASKO–SHELL [4] have proved the existence of equilibrium for an Overlapping generations model with a complete structure of markets. SCHMACHTENBERG [24] has proved the existence of equilibrium for a classical Overlapping Generations Model, in which each agent is alive for only two periods, with incomplete markets of nominal assets. SEGHIR [25] proved the existence of equilibrium in an Overlapping Generations Model, in which each agent is alive for a finite number of periods (eventually more than two), with incomplete markets of one-period numeraire assets. The second way to extend an economy to an infinite horizon is to assume that there is a finite number of eternal agents. This approach was studied by BEWLEY [7] (for complete markets) and MAGILL–QUINZII [19], LEVINE–ZAME [l8], HERNANDEZ–SANTOS [15], SANTOS–WOODFORD [23], FLORENZANO–GOURDEL [11] and ARAUJO–PÁSCOA–TORRES-MARTÍNEZ [3] (for incomplete markets).

This paper addresses the equilibrium existence in an infinite horizon incomplete market model, with one-period numeraire assets, in which the demographic structure consists of an hybrid of infinitely-lived and finitely-lived agents and the stochastic structure is given by an infinite event-tree in which each node has a finite number of immediate successors.

This demographic structure, used also by WILSON [27] for a complete structure of markets and by FLORENZANO–GOURDEL–PÁSCOA [12] for an incomplete structure of markets with real assets and bounded short-sales, is of interest for several reasons. First, these infinitely lived agents can be viewed as altruistic consider the welfare of all their dynasties. Second, as in MULLER and WOODFORD [21], one may suppose that certain institutions are effectively agents with infinite-horizon consumption programs. For example, THOMPSON [26] argues that corporations should be modelled as infinitely lived agents while private households are finitely lived. Third, one may interpret the Overlapping generations model as a representation of the behavior of agents who optimize over a finite horizon, not because of their biological life span, but because of financial constraints (for example, in SEGHIR [25], the agents are constrained to attend the financial market at all the periods of their lifetime except at the last one). In the present model, one may interpret the demographic structure as a one in which some agents are financially constrained and other are not.

As well known, the first problem which appears in presence of infinitely-lived agents is the possibility of Ponzi schemes (see LEVINE [l7] for more details). This problem does not exist for finitely lived agents. Indeed, for a finitely lived agent, the fact that he can not attend the financial market at the last period of his lifetime, with the budget constraints, rules out this problem. In contrast, even if the system of prices does not offer arbitrage opportunities, the decision problem of an infinitely lived agent, may not have a solution. Indeed, for any no-arbitrage price system, an infinitely lived agent can obtain one more unit of income at some node and can roll over his debt ad infinitum thereafter. This process permits to him to renew his credit and postpone the repayment of his debts until infinity. If the preferences of agents are monotone, this new consumption-portfolio plan will be preferred. So, in order to have a solution for the problems of the infinitely lived agents, we must impose some mechanisms for them that avoid such process. Several approach were used to rule out such Ponzi schemes. The first approach, known as *short-sales constraints*, imposes that the borrowed quantities are uniformly bounded from above along the event-tree and therefore guarantees that the agents can not resort to Ponzi schemes, because an agent who would like to do so would, sooner or later, exceed the bound a priori imposed on his short-sales. When markets are complete, Ponzi schemes can be prevented also by imposing that the borrowed values should be less than the current value of the initial endowments: *solvency requirement* (see Bewley [7] for a model with a finite number of infinitely lived agents and Wilson [27] for a model which covers the case of a countable set of infinitely lived agents, and also an hybrid of infinitely lived and finitely lived agents). Now, as is known, when markets are incomplete, the current value prices become personalized (the agents can differ about this current value price) and, since the markets are anonymous, this causes some complications about solvency requirement. As well explained by Levine–Zame [[l8], p. 104], an alternative formulation of the solvency requirement when markets are complete, is that at each node of the event-tree, almost all the debt can be reimbursed in a finite number of periods: *finitely effective debt constraints.* Levine–Zame [l8] showed that this requirement has in fact the same intuition than the usual *solvency condition* and also avoids Ponzi schemes. Levine–Zame [l8] impose alternatively some *debt constraints* (to the eternal agents) which are compatible with some equilibrium and, as proved in their paper, are more general than finitely effective debt constraints. More precisely, Levine–Zame [l8] require that at each node of the event-tree, the possible amount of the debt of each infinitely lived agent is bounded from below by a system of debt constraints (which depends on the nodes). This requirement is weaker than the short-sales constraints since the system of debt constraints is less restrictive than the short-selling constraints. Indeed, we can always associate to a portfolio satisfying short-sales constraints a system of debt constraints which will be equal to the uniform bound (a priori) imposed on the short-sales multiplied by the return of the asset. The system of debt constraints which will result from this operation is not uniform on the even-tree but depends on the nodes. In fact, the system of debt constraints as defined by Levine–Zame [l8] are not uniform on the event-tree and can explode asymptotically. To rule out this possible explosion of the debts and to make them compatible with some equilibrium, Levine–Zame [l8] were interested only by the systems of debts satisfying two conditions which depend on the considered commodity-asset price system. The first one is the looseness: if there exists a portfolio which generates a debt satisfying tomorrow's debt constraints then this debt satisfies the today's debt constraint. The second condition is the consistency: if the debt satisfies today's debt constraints then there exists a portfolio which refinances this debt satisfying tomorrow's debt constraints. For example, finitely effective debt constraints are loose and consistent and, as shown by Levine–Zame [l8], a system of debt constraints which is loose and consistent *and is compatible with some equilibrium* is necessarily finitely effective: that is these two equilibrium concepts coincide. To avoid Ponzi schemes, Magill–Quinzii [19] use *a transversality condition* which limits the rate of growth of debt and prevents the asymptotic explosion of the debts. In the deterministic case (i.e.: each node of the event-tree has a unique immediate successor), this condition requires that debts grow asymptotically slower than the rate of interest (see, for example, Koehe [16] and Blanchard–Fischer [8]). Magill–

Quinzii [19] propose an equilibrium concept based on personalized present value, but Levine–Zame [18] prove that the equilibrium notion of Magill–Quinzii does not differ from the notion of equilibrium with loose and consistent debt constraints. Moreover, Florenzano–Gourdel [11] show that, under suitable assumptions, these two notions are equivalent in the nominal case.

For the same demographic structure than the one used in this paper and for an incomplete structure of financial markets with real assets, Florenzano–Gourdel–Páscoa [12] proved the existence of equilibrium for a model in which the preferences of the agents are supposed to be represented by strictly increasing time- and state-separable utility function. Since when the assets are real, the equilibrium may fail to exist if the short sales are not bounded a priori[1], Florenzano–Gourdel–Páscoa [12] imposed short-sales constraints. Moreover, since the stochastic structure considered by Florenzano–Gourdel–Páscoa [12] is characterized by a continuum of successors at each node, they assumed harsh short-selling constraints in order to guarantee that ex-post revenues are positive and bounded away from zero.

In this paper, I prove the existence of equilibrium for a model which has the same demographic structure than Wilson [27] and Florenzano–Gourdel–Páscoa [12] when the preferences of the agents are represented by weakly monotone preorders. Since I consider incomplete markets with *numeraire assets*, I do not need to impose short-sales constraints made by Florenzano–Gourdel–Páscoa [12], mainly because it turns out that, in the numeraire case, the rank of the return matrix does not drop at equilibrium. As will be explained more precisely later on, this is guaranteed by the fact that the value of the numeraire, at equilibrium, is non-zero at each node of the event-tree. Moreover, since the stochastic structure I consider in this model is characterized by a finite number of immediate successors at each node, the positivity of the ex-post incomes, needed in Florenzano–Gourdel–Páscoa [12], is not required in this paper. However, to rule out Ponzi schemes for eternal agents, I use systems of debts constraints due to Levine–Zame [18]. Nevertheless, one could show the existence of equilibrium for our model using a transversality condition[2].

When default is allowed, Araujo–Páscoa–Torres-Martínez [3] use *collateral requirement* to show the existence of equilibrium without imposing neither debt constraints nor transversality conditions, but assuming that the preferences of the agents are represented by time- and state-separable utility functions. The obligation of constituting collateral in terms of durable goods whenever an asset is sold will limit the asymptotic explosion of the debt since, in their model, the collateral is required to be different from zero for all assets negotiated in the economy and the aggregate endowment of these durable goods is bounded.

Aside the introduction of some mechanism which avoids Ponzi schemes (namely debt constraints in this paper), the existence of eternal agents forces us to add some assumptions, in comparison with a classical Overlapping Generations Model. The first one is the Mackey continuity of the preferences of eternal agents which guar-

1. Because the rank of the return matrix may drop (see Hart [14]). In fact, as is known, if such short-sales constraints are not assumed, one can obtain only a pseudo-equilibrium.
2. The proof made by Magill–Quinzii [19] for a model with a finite number of agents has to be carefully redone in our model; in which the number of agents at each node is finite but varies from one node to another and the set of agents is countable. Indeed, to prove the equilibrium existence with a transversality condition, Magill–Quinzii [19] proved that the value of the debt of each infinitely lived agent is uniformly bounded. This was mainly guaranteed by the fact that the number of agents of the whole economy is finite. Since in our model, the number of agents changes from one node to another, this proof of uniform bounds on the debt values no longer holds and must be remade.

antees the existence of a degree of impatience for each agent with respect to his consumptions (for more details see MAGILL–QUINZII [19]). The second additional assumption requires that for the eternal agents, this proportion is bounded away from zero uniformly across the nodes. Note that SEGHIR [25] has proved that, under a usual continuity assumption of preferences in finite dimension, this proportion depends on agents and is uniform on nodes for finitely lived agents. The third additional assumption is the existence of a uniform upper bound of the initial aggregate dotations.

To prove the existence of equilibrium, I first truncate the infinite horizon economy at some period T. Using SEGHIR [25], each truncated (finite horizon) economy has an equilibrium (with no debt constraints other the fact that in the period T, the agents can not attend the financial market). For each infinitely lived agent, I associate to this equilibrium an implicit system of debt constraints which is loose and consistent with respect to commodity-asset price equilibria of the truncated economies. Then, passing to the limit on T, I prove that the limit of these finite horizon equilibria is an equilibrium of the infinite horizon economy. In this equilibrium, the debt constraints are the limits of implicit debt constraints of the truncated economies. Finally, I prove that the system of debt constraints of the initial economy is loose and consistent.

This paper is organized as follows: In Section 2, I present the model. In Section 3, I state the assumptions and the existence equilibrium result. Section 4 deals with the truncated economies and the existence of equilibrium for each one. Section 5 shows that, passing to the limit, one has an equilibrium with debt constraints for the original economy. Moreover, in this section, I prove that the system of debt constraints, associated to the equilibrium, is loose and consistent. Section 6 concludes the paper. Appendix A is devoted to some technical proofs and in Appendix B, I prove that, on the bounded subsets of ℓ_∞, the Mackey topology $\tau(\ell_\infty, \ell_1)$ and the weakstar topology $\sigma(\ell_\infty, \ell_1)$ coincide, a result needed mainly in Section 4.

2 The Model

I consider an infinite horizon economy for which the stochastic structure is described by an infinite tree with an unique root. Formally, let $\mathcal{T} = \{0, 1,...\}$ be the set of periods and let S be the set of states of nature. The revelation of information is described by a sequence of partitions of S, $(\mathbf{F}_0, \mathbf{F}_1, \ldots ,\mathbf{F}_t, \ldots)$, where the number of subsets in $\mathbf{F}_t$ is finite and $\mathbf{F}_{t+1}$ is finer than the partition $\mathbf{F}_t$ (i.e.: $\sigma \in \mathbf{F}_{t+1}$, $\sigma' \in \mathbf{F}_t \Rightarrow \sigma \subset \sigma'$ or $\sigma \cap \sigma' = ?$) $\forall t \geqslant 0$.

At node 0, I assume that there is no information so that $\mathbf{F}_0 = S$. The information available at time t (for all $t \in \mathcal{T}$) is assumed to be the same for all agents in the economy (symmetric information) and described by the subset σ of the partition $\mathbf{F}_t$ in which the state of nature lies.

A pair $\xi = (t, \sigma)$ where $t \in \mathcal{T}$ and $\sigma \in F_t$ is called node and $t(\xi) = t$ is the date of node ξ. The set D consisting of all nodes is called the event-tree introduced by $\mathbf{F}$ **:**

$$D = \bigcup_{\substack{t\in\mathcal{T}\\ \sigma\in F_t}} \{(t,\sigma)\}.$$

A node $\xi' = (t', \sigma')$ is said to succeed (resp. strictly) node $\xi = (t, \sigma)$ if $t' \geqslant t$ (resp. $t' > t$) and $\sigma' \subset \sigma$. I write $\xi' \geqslant \xi$ (resp. $\xi' > \xi$). The set of nodes which succeed a node $\xi \in D$ is called the subtree $D(\xi)$ and $D^+(\xi) = \{\xi' \in D \mid \xi' > \xi\}$ is the set of strict successors of ξ. The subset of nodes of $D(\xi)$ at date T is denoted by $D_T(\xi)$ and the subset of nodes between $t(\xi)$ and T by $D^T(\xi)$; $D^T = \{\xi \in D \mid t(\xi) \leqslant T\}$.

When ξ is the initial node, the notations are simplified to D^+, D_T, D^T.

For each $\xi \in D$, $\xi^+ = \{\xi' \in D(\xi) \mid t(\xi') = t(\xi) + 1\}$ is the set of immediate successors of ξ. The number of elements of ξ^+ is finite and is called the branching number $b(\xi)$ at $\xi (b(\xi) = \#\xi^+)$.

If $\xi = (t, \sigma)$, $t \geqslant 1$, the unique node $\xi^- = (t - 1, \sigma')$, $\sigma \subset \sigma'$ is called the predecessor of ξ.

At each node $\xi \in D$, a finite number G of physical goods, indexed by $g = 1, ..., G$, are traded on spot markets by alive consumers and a finite number $J(\xi)$ of one-period numeraire assets are available for intertemporal transaction and insurance to any consumer who is currently alive and will be alive also at the following period. The set consisting of all commodities indexed over the event-tree is thus:

$$D \times G = \{(\xi, g) : \xi \in D,\ g = 1, \ldots, G\}.$$

Let $\mathbf{R}^{D \times G}$ denote the vector space of all maps $x : D \times G \to \mathbf{R}$ and let $\ell_\infty(D \times G)$ denote the subspace of $\mathbf{R}^{D \times G}$ consisting of all bounded maps (sequences); i.e.: $\ell_\infty(D \times G) := \{x \in \mathbf{R}^{D \times G} : \sup_{(\xi,\, g) \in D \times G} |x(\xi,\, g)| < \infty\}$. The norm $\|\cdot\|_\infty$ of $\ell_\infty(D \times G)$ is defined by $\|x\|_\infty = \sup_{(\xi,\, g) \in D \times G} |x(\xi,\, g)|$.

I take the commodity space to be $\ell_\infty(D \times G)$.

Let a consumption bundle $e \in \mathbf{R}^G_+$ be chosen as a numeraire.

Since all assets are numeraires, at each $\xi \in D$, $r^j(\xi) \in \mathbf{R}^+$ denotes the return of asset j (bought at the node ξ^-), denominated in units of numeraire. I denote $r(\xi) = (r^j(\xi))_{j \in J(\xi^-)}$, $R(\xi) = (r^j(\xi'))_{\substack{j \in J(\xi) \\ \xi' \in \xi^+}}$ and $R := \prod_{\xi \in D} R(\xi)$. The numeraire e and the $(b(\xi) \times \#J(\xi))$-matrix $R(\xi)$ summarize the numeraire asset structure at node ξ.

Let $p(\xi) = (p(\xi, g), g \in G)$ denotes the vector of spot prices for the G goods at node ξ and $p = (p(\xi), \xi \in D) \in \mathbf{R}^{D \times G}$ be the spot price process.

Let $q(\xi) = (q(\xi, j), j \in J(\xi)) \in \mathbf{R}^{J(\xi)}$ be the vector of prices of the securities issued at node ξ and let $q = (q(\xi), \xi \in D)$ denote the security price process which belongs to security price space $\prod_{\xi \in D} \mathbf{R}^{J(\xi)}$.

As mentioned before, the demographic structure of the model includes eventually both an overlapping generations of finite lived agents and a finite number of infinite lived agents. For simplicity, I will assume, that each infinite lived agent is born at the initial node. Commodity endowments and preferences are affected by the uncertainty but the birth and the death of agents are not affected by the uncertainty.

Let I be the set of agents, which is countable. Each agent $i \in I$ is characterized by:

- A consumption set $X^i \subset \mathbf{R}^{D \times G}$,
- A portfolio set $Z^i \subset \prod_{\xi \in D} \mathbf{R}^{J(\xi)}$,

- A preorder of preferences $\succeq^i$ on X^i,
- An initial endowment $\omega^i \in X^i$,
- The collection of his lifetime periods $T^i \subset \{0, 1,...\}$. Without loss of generality, one can assume that for each agent $i \in I$, T^i is a finite or an infinite interval of $\mathbf{N}$.
- The collection of periods $T'^i \subset \{0, 1,...\}$ where agent i can attend the financial market.

Let us define for each agent $i \in I$, the correspondence preferences $P^i : X \to X^i$ by: $\forall x \in X, P^i(x) := \{x'^i \in X^i \mid x'^i \succ^i x^i\}$.
I define the economy $\mathcal{E}$ as follows:

$$\mathcal{E} = ((X^i, \succeq^i, \omega^i, Z^i, T^i, T'^i)_{i \in I}; R; e).$$

3 Assumptions and Result

Let $\ell_1(D \times G)$ denote the subspace of $\mathbf{R}^{D \times G}$ consisting of all summable sequences; i.e.: $\ell_1(D \times G) := \{p \in \mathbf{R}^{D \times G} : \sum_{(\xi, g) \in D \times G} |p(\xi, g)| < \infty\}$.
For $p \in \ell_1(D \times G)$ and $x \in \ell_\infty(D \times G)$, $p \cdot x$ will denote:

$$p \cdot x = \sum_{(\xi, g) \in D \times G} p(\xi, g)\, x(\xi, g).$$

The Mackey topology, $\tau(\ell_\infty(D \times G), \ell_1(D \times G))$, on $\ell_\infty(D \times G)$ is the strongest locally convex topology consistent with the duality between $\ell_\infty(D \times G)$ and $\ell_1(D \times G)$ (for more details see p. 219 in Aliprantis–Border [1]).
I make on $\mathcal{E}$ the following assumptions:

A.1: $\forall \xi \in D$, $b(\xi) < +\infty$ and $1 \leqslant \#J(\xi) < \infty$.

Assumption (A.1) is classical in a such model. To prove the existence of equilibrium with unbounded short sales, it is essential that each node has a finite number of immediate successors. Moreover, (A.1) requires that at each node, the number of assets is finite.
I define the following subsets of I,

$$I_1 = \{i \in I \ : \ T^i \text{ is finite}\},$$

$$I_2 = \{i \in I \ : \ T^i \text{ is infinite}\}.$$

Remark 3.1: If the set I_2 is empty, this model is equivalent to an Overlapping Generations model of finitely lived agents (see Seghir [25]). If the set I_1 is empty and the set I_2 is finite, the demographic structure of the model is equivalent to the one considered by Magill–Quinzii [19], Levine–Zame [18], Florenzano–Gourdel [11] and Araujo–Páscoa–Torres-Martínez [3].

A.2: • For each agent $i \in I$, T^i is a finite interval of **N** or is equal to **N**.
• If $T^i = \mathbf{N}$, $T'^i = \mathbf{N}$.
• If $T^i = \{m, m+1, ..., n\}$, $T'^i = \{m, m+1, ..., n-1\}$.

A.3: At each period $t \in \mathcal{T}$, one has:
(i) $H(t) = \{i \in I \mid t \in T^i\}$ is finite.
(ii) There exists an agent i_t such that $t \in T'^{i_t}$.

The condition (i) of Assumption (A.3) imposes that at each period there is a finite number of alive agents. The condition (ii) of Assumption (A.3) imposes that at each period there is at least one agent who can attend the financial market. In fact, it guarantees that the entire economy is sufficiently connected. This condition is made purely to ease and with no loss of generality. Indeed, if this requirement (ii) is not satisfied, one can replace the economy by a collection of finite-horizon economies and at most one infinite horizon economy which satisfies Condition (ii).

Remark 3.2: The condition (ii) of Assumption (A.3) is obviously satisfied for an overlapping generations model in which, as in SCHMACHTENBERG [24], each agent is alive for only two periods and at each period, there is at least one agent who is born. This condition is also satisfied if the set I_2 is nonempty. Under the condition (ii) of Assumption (A.3), the set H(t) is obviously nonempty for each period $t \in \mathcal{T}$. Assumptions (A.2) and (A.3) imply that the set I_2 is finite.

A.4: $\forall i \in I, \forall \xi \in D,\ X^i(\xi) = \mathbf{R}^G_+$ if $t(\xi) \in T^i$ and $X^i(\xi) = \{0\}$ if $t(\xi) \notin T^i$.
For each $i \in I,\ X^i = (\prod_{\xi : t(\xi) \in T^i} X^i(\xi)) \cap \ell^+_\infty(D \times G)$.
For each agent i in I_1, $\omega^i(\xi, g) > 0$, $\forall \xi : t(\xi) \in T^i$, $\forall g \in G$.
Moreover,
(i) There exists M > 0 such that $\sum_{i \in I} \omega^i(\xi, g) < M,\ \forall \xi \in D,\ \forall g \in G$,
(ii) There exists $M' > 0$ such that $\forall i \in I_2$, $\forall \xi \in D$, $\forall g \in G$, $\omega^i(\xi, g) > M'$.

Assumption [A.4] interprets in particular the natural condition that the agents consume only when they are alive and requires that each agent has a positive initial endowment at each period of his lifetime duration. The conditions (i) and (ii) are classical in infinite horizon models and require that the aggregate initial endowments are uniformly bounded from above and that the initial endowment of each alive agent is bounded away from zero.

A.5: $\forall i \in I$, $\forall \xi \in D$, $Z^i(\xi) = \mathbf{R}^{J(\xi)}$ if $t(\xi) \in T'^i$ and $Z^i(\xi) = \{0\}$ if $t(\xi) \notin T'^I$.

Assumption [A.5] requires that the agents can attend the financial market at each period of their lifetime except the last one. Let us notice that [A.5] does not impose any short-sales constraints at these periods.

A.6: $\forall i \in I$, $\succeq^i$ is a complete and monotone preorder which verifies that for every $\lambda \in [0, 1[$, for every $i \in I$ and for every $x'^i \succ^i x^i$ one has $\lambda x^i + (1-\lambda)x'^i \succ^i x^i$. Moreover, $\forall x^i \in X^i$, $\{\tilde{x}^i \in X^i \mid \tilde{x}^i \succeq^i x^i\}$ is convex and closed for the Mackey topology (hence closed for the weak topology) induced on X^i and $\{\tilde{x}^i \in X^i \mid \tilde{x}^i \succ^i x^i\}$ is open[3] in X^i for the Mackey topology induced on X^i.

3. Note that it follows from Assumption (A.6) that the graph of P^i is open (see Theorem 3 in BERGSTROM–PARKS–RADER [5]).

As shown in Bewley [7], $\succeq^i$ is Mackey continuous if it is represented by an additively separable utility function. Note that for finitely lived agents, the Mackey continuity assumed in A.6 is equivalent to the usual continuity in a finite dimensional commodities space.
Let us define the feasible consumption set by:

$$\hat{X} = \{(x^i)_{i\in I} \in \prod_{i\in I} X^i \mid \forall(\xi,g) \in D\times G,\ \sum_{i\in I} x^i(\xi,g) = \sum_{i\in I} \omega^i(\xi,g)\}.$$

Note that $\hat{X}$ is closed for the product topology (on i) of the $\sigma(\ell_\infty(D \times G), \ell_1(D \times G))$ topologies. For each $i \in I$, I denote by $\hat{X}^i$ the projection of $\hat{X}$ on X^i. Note that, for each $i \in I$, $\hat{X}^i$ is $\sigma(\ell_\infty(D \times G), \ell_1(D \times G))$–compact.

A.7: (Desirability of numeraire for each agent at each node of his lifetime periods and every component of feasible consumption).

$\forall i \in I,\ \forall x \in \hat{X},\ \forall \xi : t(\xi) \in T^i,\ \exists\, \lambda \in]0,1]$ such that $x^i + \lambda e^\xi \succ^i x^i$ where e^ξ is defined by: $e^\xi(\xi) = e$ and $e^\xi(\xi') = 0$, $\forall \xi' \neq \xi$.
As we will see later, under (A.7), the Mackey continuity, assumed in (A.6), implies for each agent the existence of a degree of impatience with respect to consumptions. This degree depends on the agent and the considered node. Since, there is a finite number of eternal agents in our model, this degree will be uniform on these agents. Moreover, for finitely lived agents, this degree will be uniform on nodes (but depends on the agent).
From now on, I adopt the following normalization: $\forall \xi \in D$, $p(\xi) \cdot e = 1$.
For each finitely lived agent $i \in I_1$, I define the budget set as follows:

Definition 3.1: *Given a system of prices* (p, q), *the budget set* $B^i(p, q)$ *of an agent* $i \in I_1$, *such that* $T^i = \{m, m + 1,..., n\}$ *is the set of* $(x^i, z^i) \in X^i \times Z^i$ *satisfying:*

$$p(\xi)\cdot(x^i(\xi) - \omega^i(\xi)) + q(\xi)\cdot z^i(\xi) \leqslant r(\xi)\cdot z^i(\xi^-),\ \forall \xi : t(\xi) \in T^i$$

and

$$z^i(\xi) = 0,\ \forall \xi : t(\xi) = n,$$

with $z^i(0^-) := 0$.

As well known, the definition of the budget sets of infinitely lived agents needs some additional condition which rules out Ponzi schemes. In this work, I use systems of debt constraints, due to Levine–Zame [18]. As mentioned at the beginning of the paper, there are other possibilities to avoid Ponzi schemes (see for example Magill–Quinzii [19], Hernandez–Santos [15] and Araujo–Páscoa–Torres-Martínez [3]). Recall that Florenzano–Gourdel [11] have proved that the approach used by Levine–Zame [18] is equivalent to the one used by Magill–Quinzii [19] (for the nominal case). As mentioned before, the equilibrium existence for our model can also be shown using a transversality condition (See Footnote 1 above).
For each infinitely lived agent $i \in I_2$, a system of debt constraints D^i is defined as a function $D^i : D \to [-\infty, 0]$. The systems of debt constraints will be a component of equilibrium for eternal agents.

The budget sets of the eternal agents are defined as follows:

DEFINITION 3.2: *Let $i \in I_2$ be an infinitely lived agent. Given a system of prices* (p, q) *and debt constraints* D^i*, the budget set* $B^i(p, q, D^i)$*, of the agent* $i \in I_2$ *is the set of all the* $(x^i, z^i) \in X^i \times Z^i$ *satisfying:* $\forall \xi \in D$

$$p(\xi)\cdot(x^i(\xi)-\omega^i(\xi))+q(\xi)\cdot z^i(\xi)\leqslant r(\xi)\cdot z^i(\xi^-),$$

$$r(\xi)\cdot z^i(\xi^-)\geqslant D^i(\xi).$$

The debt constraints require that at each node $\xi \in D$ and for each infinitely-lived agent $i \in I_2$, the possible amount of the net debts, $r(\xi)z^i(\xi^-) := r(\xi)[\theta^i(\xi^-) - \varphi^i(\xi^-)]$, induced by his trades at the predecessor ξ^- are bounded from below by a system $D^i(\xi)$. This condition will limit the set of portfolios that the eternal agents can trade. Moreover, it implies in particular that at each node $\xi \in D$, and for each infinitely-lived agent $i \in I_2$, the debts, $r(\xi)\,\varphi^i(\xi^-)$, induced by the sales of i at node ξ^- are bounded from above by a system depending on the node ξ, namely $-D^i(\xi)$. Note that the debt constraints could be equal to $-\infty$ and obviously such debt constraints are not compatible with any equilibrium. However, we will focus, as LEVINE–ZAME [18], on debt constraints which satisfies two conditions; namely the looseness and the consistency. As we will see later on, such debt constraints will be compatible with some equilibrium.

DEFINITION 3.3: *Given a system of prices* (p, q)*, the system of debt constraints* D^i*, of an infinitely lived agent* $i \in I_2$*, is* (p, q)*–consistent at node* ξ *if there exists a portfolio* $z^i \in \mathbf{R}^{J(\xi)}$ *such that*

$$\begin{cases} D^i(\xi)+p(\xi)\cdot\omega^i(\xi)-q(\xi)\cdot z^i(\xi)\geqslant 0, \\ \text{and } r(\xi')\cdot z^i(\xi)\geqslant D^i(\xi'), \qquad \forall \xi' \in \xi^+. \end{cases}$$

The system of debt constraints D^i *is said to be* (p, q)*–consistent if it is* (p, q)*–consistent at each node.*

REMARK 3.3: For a node $\xi \neq 0$, an equivalent formulation of the previous definition is that for all $z^i(\xi^-) \in \mathbf{R}^{J(\xi^-)}$ such that $r(\xi)\cdot z^i(\xi^-) \geqslant D^i(\xi)$, there exists a portfolio $z^i(\xi) \in \mathbf{R}^{J(\xi)}$ such that:

$$\begin{cases} r(\xi)\cdot z^i(\xi^-)+p(\xi)\cdot\omega^i(\xi)-q(\xi)\cdot z^i(\xi)\geqslant 0 \\ \text{and } r(\xi')\cdot z^i(\xi)\geqslant D^i(\xi'), \qquad \forall \xi' \in \xi^+. \end{cases}$$

In other words, if an agent i arrives at a node ξ with a debt $r(\xi)\cdot z^i(\xi^-)$ satisfying the budget constraint at node ξ, then the agent i, consuming nothing today, can acquire a portfolio plan which satisfies the tomorrow's debt constraints.

DEFINITION 3.4: *Given a system of prices* (p, q)*, the system of debt constraints* D^i*, of an infinitely lived agent* $i \in I_2$*, is* (p, q)*–loose at node* ξ *if for every portfolio* $z^i(\xi) \in \mathbf{R}^{J(\xi)}$ *such that* $\forall \xi' \in \xi^+$, $r(\xi')\cdot z^i(\xi) \geqslant D^i(\xi')$*, one has* $p(\xi)\cdot\omega^i(\xi) + D^i(\xi) - q(\xi)\cdot z^i(\xi) \leqslant 0$.

The system of debt constraints D^i *is said to be* (p, q)*–loose if it is* (p, q)*–loose at each node.*

Remark 3.4: In view of the last definition, if D^i is (p, q)–loose at a node ξ then $D^i(\xi) \leqslant - p(\xi) \cdot \omega^i(\xi)$.

Definition 3.5: *An equilibrium with debt constraints of $\mathcal{E}$ is a collection* $(\overline{p},\overline{q},(\overline{x}^i,\overline{z}^i)_{i\in I},(\overline{D}^i)_{i\in I_2})$ *such that:*

(i) *For each* $i \in I_1$, $(\overline{x}^i,\overline{z}^i)$ *is optimal in* $B^i(\overline{p},\overline{q})$ *and for each* $i \in I_2$, $(\overline{x}^i,\overline{z}^i)$ *is optimal in* $B^i(\overline{p},\overline{q},\overline{D}^i)$.

(ii) *For each* $i \in I_2$, $\overline{D}^i$ is $(\overline{p},\overline{q})$ *– loose and consistent.*

(iii) *For each node* $\xi \in D, \sum_{i\in I}(\overline{x}^i(\xi) - \omega^i(\xi)) = 0,$

(iv) *For each node* $\xi \in D, \sum_{i\in I}\overline{z}^i(\xi) = 0.$

To prove the existence of equilibrium in this model, I need the following additional assumptions:

A.8: $\forall \xi \in D$, Rank $R(\xi) = J(\xi)$; there is no redundant asset[4].

Let $E \subset D,$ be a subset of nodes and let χ_E denotes the characteristic function of E. For each process x on D, I define

$$(x\chi_E)(\xi) = \begin{cases} x(\xi) & \text{if } \xi \in E \\ 0 & \text{if } \xi \notin E \end{cases}.$$

A.9: $\exists \beta < 1 : \forall i \in I_2,$

$$x^i\chi_{D\setminus D^+(\xi)} + \beta x^i \chi_{D^+(\xi)} + e^{\xi} \succ {}^i x^i,\ \forall \xi \in D,\ \forall x^i \in \hat{X}^i.$$

Assumption (A.9) reinforces, for eternal agents, the Mackey continuity assumed in (A.6). It imposes that the proportion of future consumption which an eternal agent is prepared to give up in exchange for an additional unit of the numeraire e at a node ξ is uniform on ξ. This requirement obstructs each infinitely lived agent from having a degree of impatience which disappears asymptotically. As proved by Magill–Quinzii [19], Assumption A.9 is satisfied if the preferences of agents are represented by an additively separable utility functions:

$$U^i(x^i) = \sum_{\xi\in D}\alpha(\xi)\,\delta_i^{t(\xi)}u^i(x^i(\xi)),$$

where $\alpha(\xi)$ is a probability, $\delta_i^{t(\xi)} \in [0,1]$ and $u^i : \mathbf{R}_+^G \to \mathbf{R}$ is a continuous, increasing concave function with $u^i(0) = 0$. Recall that under the continuity assumption

4. Once the existence of an equilibrium has been shown for a non redundant set of assets, the equilibrium is not affected by the introduction of redundant assets. Moreover, Assumption [A.8] guarantees that at each node $\xi \in D$, $J(\xi) \leqslant b(\xi)$. That is, at each node of the event-tree, there is at most as many assets as immediate successors.

made in A.6 and the desirability of numeraire made in A.7, we have the following result for finitely lived agents:

LEMMA 3.1: (SEGHIR [25]): $\forall i \in I_1$, $\exists \beta^i < 1$ *such that:*

$$x^i \chi_{D \setminus D^+(\xi)} + \beta^i x^i \chi_{D^+(\xi)} + e^\xi \succ^i x^i,\ \forall \xi\ :\ t(\xi) \in T^i,\ \forall x^i \in \hat{X}^i.$$

A.10: (Non-risky asset) For each ξ in D, there exists j_ξ in $J(\xi)$ such that $r^{j_\xi}(\xi') = 1, \forall \xi' \in \xi^+$.

The existence, at each node, of an asset which generates a positive return at each immediate successor is classical for such a model. For example, an indexed bond verifies Assumption (A.10). Note that such an assumption does not interfere neither in the finite horizon models nor in classical Overlapping generations models. That is, if the set I_2 is empty then Assumption (A.10) is unnecessary. Moreover, under (A.10), the positivity of returns is unnecessary to prove the existence of equilibrium. More precisely, let us consider any return structure $R' := (R'_\xi)_{\xi \in D}$. Obviously, there exists a collection $(R_\xi)_{\xi \in D}$ such that for all $\xi \in D$, $R_\xi \geqslant 0$ and $\operatorname{Im} R_\xi = \operatorname{Im} R'_\xi$. It is easy to see that an equilibrium with the return structure R can be deduced from an equilibrium with the return structure R'.

THEOREM 3.2: *Under Assumptions (A.1)-(A.10) stated above, the economy $\mathcal{E}$ has an equilibrium* $(\overline{p}, \overline{q}, (\overline{x}^i, \overline{z}^i)_{i \in I}, (\overline{D}^i)_{i \in I_2})$ *which satisfies:* $\forall \xi \in D,\ \overline{p}(\xi) \cdot e > 0$.

4 The Truncated Economy

In this section, we define truncated economies which are finite horizon economies. We prove that each truncated economy has an equilibrium. We construct then some suitable implicit debt systems for the truncated economies and we show then that the components (including the implicit debt systems) of each equilibrium of the defined truncated economies are bounded in order to guarantee the existence of convergent subsequences.

4.1 Definitions

Let $\mathcal{E}^T$ be the truncated economy associated to the original $\mathcal{E}$, which has the same characteristics than $\mathcal{E}$, but where I suppose that agents are constrained to stop their exchange of goods at period T and their exchange of assets at period $T-1$. More precisely, I define the following sets:

- $I^T = \{i \in I \mid \exists t \leqslant T,\ t \in T^i\} = \bigcup_{t \leqslant T} H(t)$.

• Given (p, q),

$$B^{iT}(p,q)=\left\{(x^i,z^i)\;\middle|\; \begin{array}{l} \forall\xi\ :\ t(\xi)\geqslant T,\ z^i(\xi)=0,\\ \forall\xi\ :\ t(\xi)>T,\ x^i(\xi)=0,\\ \forall\xi\in D^T,\\ p(\xi)\cdot(x^i(\xi)-\omega^i(\xi))+q(\xi)\cdot z^i(\xi)\leqslant r(\xi)\cdot z^i(\xi^-). \end{array}\right\}.$$

Remark 4.1: the truncated economy $\mathcal{E}^T$ is a standard (T + 1)–periods incomplete markets model of one-period numeraire assets with a finite number of agents.

According to [Proposition 4.2, Seghir [25]], under Assumptions (A.1)-(A.8), each truncated economy $\mathcal{E}^T$ has an equilibrium $(\overline{p}^T, \overline{q}^T, \overline{x}^T, \overline{z}^T)$ which verifies $\forall\xi\in D^T,\ \overline{p}^T(\xi)\cdot e>0.$
Let us recall also the following lemma:

Lemma 4.1: [Lemma 5.1, Seghir [25]].

Under Assumptions $(A.1)-(A.7)$, $\forall i\in I^T$, $\exists\,\overline{\pi}_T^i\in\mathbf{R}^{D^T}$ *such that:*

(i): $\overline{\pi}_T^i(\xi)>0,\ \forall\xi\ :\ t(\xi)\in T^i,$

(ii): $\overline{x}^{iT}$ *is optimal in* $B^{iT}(\overline{P}_T^i,\omega^i)=\{x^i\in X^i\mid \overline{P}_T^i\cdot(x^i-\omega^i)\leqslant 0\}$, *where* $\overline{P}_T^i=(\overline{P}_T^i(\xi),\ \xi\in D^T):=(\overline{\pi}_T^i(\xi)\overline{p}^T(\xi),\ \xi\in D^T),$

(iii): $\overline{\pi}_T^i(\xi)\,\overline{q}^T(\xi,j)\ =\ \sum_{\xi'\in\xi^+}\overline{\pi}_T^i(\xi')\,r^j(\xi'),\ \forall j\in J(\xi),\ \forall\xi\ :\ t(\xi)\in T'^i,$

(iv): $\sum_{(\xi,g)\in\mathbf{R}^{D^T\times G}}\overline{\pi}_T^i(\xi)\overline{p}^T(\xi,g)=1.$

For each $i\in I^T$, $\overline{\pi}_T^i$ denotes the present value vector and $\overline{P}_T^i$ denotes the vector of discounted prices.

Remark 4.2: Magill-Quinzii [19] have used a similar result to Lemma 4.1: but in their model, each agent is alive at every period of the economy. That is, in their model, each agent has a positive initial endowment at each period of the economy.

Remark 4.3: The conditions (i) and (ii), in Lemma 4.1, express that the first order conditions for the problem of maximization of agent i are satisfied at $(\overline{x}^i,\overline{z}^i)$ if $\overline{\pi}_T^i$ is the multiplier associated with the budget constraint at node ξ for each ξ.

Moreover, the condition (ii) expresses that the discounted price vector $\overline{P}_T^i$ supports the preferred set of agent i at $\overline{x}^i$. The condition (iii) expresses the first order condition with respect to $\overline{z}^i$.

For each node $\xi \in D$, let us define the set $I(\xi)$, of the agents alive at the node ξ, as follows:

$$I(\xi) := \{i \in I \mid t(\xi) \in T^i\}.$$

Let us introduce the following lemma which will be used to get bounds on the equilibrium commodity price vector:

LEMMA 4.2: $\forall \xi \in D, \exists\, \alpha_\xi \in]0,1[: \forall i \in I(\xi)$,

$$\alpha_\xi x^i + e^\xi \succ^i x^i, \ \forall x^i \in \hat{X}^i.$$

See Appendix A for a proof.

In fact, Lemma 4.2 states that at each node $\xi \in D$, all alive agents at this node are prepared to give up some proportion $(1 - \alpha_\xi)$ of any attainable allocation $x^i \in \hat{X}^i$ in exchange for one unit of the numeraire e at the node ξ.

4.2 **Compactness Arguments**

In this part, and in order to guarantee the existence of a convergent subsequence of the equilibrium collection, we prove that each component of the truncated economy equilibrium is bounded.
Let us fix $\xi \in D^T$.

– *Bounds on* $\overline{p}^T(\xi, g)$:

By Lemma 4.1, $\forall i \in I(\xi)$, $\overline{x}^{iT}$ is optimal in $B^{iT}(\overline{P}_T^i, \omega^i)$. Moreover, by Lemma 4.2, $\exists\, \alpha_\xi \in]0,1]$ such that $\forall i \in I(\xi), \alpha_\xi \overline{x}^{iT} + e^\xi \succ^i \overline{x}^{iT}$. Hence, $\forall i \in I(\xi)$, $\overline{P}_T^i \cdot (\alpha_\xi \overline{x}^{iT} + e^\xi) > \overline{P}_T^i \cdot \overline{x}^{iT}$. Then,

$$\forall i \in I(\xi),\ \overline{\pi}_T^i(\xi)\,\overline{p}^T(\xi)\cdot e > (1-\alpha_\xi)\,\overline{P}_T^i \cdot \overline{x}^{iT} = (1-\alpha_\xi)\overline{P}_T^i \cdot \omega^i.$$

By Assumption (A.4), there exists $M' > 0$ such that $\forall i \in I_2$, $\forall g \in G$, $\omega^i(\xi, g) > M'$. Then, $\forall i \in I_2$, $\overline{\pi}_T^i(\xi) > (1-\alpha_\xi)M'$.
Moreover, $0 \leqslant \overline{P}_T^i(\xi, g) = \overline{\pi}_T^i(\xi)\overline{p}^T(\xi, g) \leqslant 1$, $\forall i \in I(\xi)$, $\forall g \in G$. Hence, $\forall i \in I_2$, $\forall g \in G$, $\overline{p}^T(\xi, g) \leqslant \dfrac{1}{\overline{\pi}_T^i(\xi)}$. Consequently,

$$\forall g \in G,\ \overline{p}^T(\xi, g) \leqslant \frac{1}{(1-\alpha_\xi)M'}.$$

– *Bounds on* $\overline{\pi}_T^i(\xi)$:

Since $0 \leqslant \overline{P}_T^i(\xi,g) := \overline{\pi}_T^i(\xi)\overline{p}^T(\xi,g) \leqslant 1,\ \forall i \in I(\xi),\ \forall g \in G,$ in view of our normalization, one gets the following bounds on $\overline{\pi}_T^i(\xi)$,

$$\forall i \in I(\xi),\ 0 \leqslant \overline{\pi}_T^i(\xi) \leqslant \| e \|_1 .$$

– *Bounds on* $\overline{q}^T(\xi, j)$:

Since $\sum_{i\in I(\xi)} \overline{z}^{iT}(\xi) = 0,$ there exists an (infinitely-lived or finitely-lived) agent $i \in I(\xi)$ such that $\overline{q}^T(\xi)\cdot \overline{z}^{iT}(\xi) \geqslant 0.$ Let us distinguish the two following cases:

• If the agent i is a finitely-lived agent (i.e.: $i \in I_1$), one gets bounds on $\overline{q}^T(\xi, j)$ as in Seghir [25].

• If the agent i is an infinitely-lived agent (i.e.: $i \in I_2$). Let us consider the following change on the portfolio of the agent i from node ξ onwards:

$$z^i = \begin{cases} \beta \overline{z}^{iT}(\xi) - \eta\,(0,\ldots,0,1,0,\ldots,0) & \text{at node } \xi \\ \beta \overline{z}^{iT}(\xi') & \forall \xi' \in D^+(\xi), \end{cases}$$

where $\eta \in \mathbf{R}^+ \backslash \{0\}$ and the component 1 in the vector (0,..., 0, 1, 0,..., 0) denotes one unit of the non-risky asset j_ξ.
Moreover, I consider the following change in the consumption of agent i from node ξ onwards:

$$x^i = \begin{cases} \overline{x}^{iT}(\xi) + e^{\xi} & \text{at node } \xi \\ \beta \overline{x}^{iT}(\xi') & \forall \xi' \in D^+(\xi) \end{cases}$$

At each node $\xi' \in D^+(\xi)\backslash\xi^+$, the budget constraints are obviously satisfied for this new consumption-portfolio plan (x^i, z^i).
At each node $\xi' \in \xi^+$, the budget constraint is satisfied by this new consumption-portfolio plan (x^i, z^i) if

$$\overline{p}^T(\xi')\cdot[\beta \overline{x}^{iT}(\xi') - \omega^i(\xi')] + \overline{q}^T(\xi')\cdot(\beta \overline{z}^{iT}(\xi')) \leqslant r(\xi')\cdot \beta \overline{z}^{iT}(\xi) - \eta.$$

Since $(\overline{x}^{iT}, \overline{z}^{iT})$ satisfies the budget constraint at each node $\xi' \in \xi^+$, this inequality is satisfied if $\eta \leqslant (1-\beta)\,\overline{p}^T(\xi')\cdot \omega^i(\xi')$. Since $\forall g \in G,\ \omega^i(\xi', g) \geqslant M'$, one has $\forall g \in G,\ \omega^i(\xi', g)\, \overline{p}^T(\xi', g)\, e^g \geqslant M' \overline{p}^T(\xi', g) e^g$. Hence, in view of our normalization, one has $\sum_{g \in G} \omega^i(\xi', g)\, \overline{p}^T(\xi', g)\, e^g \geqslant M'$. Moreover, since $\sum_{g \in G} \omega^i(\xi', g)\, \overline{p}^T(\xi', g)\, e^g \leqslant \| e \|_1 \sum_{g \in G} \omega^i(\xi', g)\, \overline{p}^T(\xi', g) = \| e \|_1\, \overline{p}^T(\xi') \cdot \omega^i(\xi')$, one has $\overline{p}^T(\xi') \cdot \omega^i(\xi') \geqslant \frac{M'}{\|e\|_1}$. Consequently, setting $\eta = \frac{(1-\beta) M'}{\|e\|_1}$, one has (x^i, z^i) satisfies the budget constraints at each node $\xi' \in \xi^+$.

However, since $\overline{x}^{iT} + e^{\xi} \succ^i \overline{x}^{iT}$ and in view of Assumption (A.9) and the optimality of $(\overline{x}^{iT}, \overline{z}^{iT})$, the budget constraint at the node ξ can not be satisfied by the new consumption-portfolio plan (x^i, z^i) i.e.:

$$\overline{p}^T(\xi) \cdot (\overline{x}^{iT}(\xi) + e - \omega^i(\xi)) + \beta \overline{q}^T(\xi) \cdot \overline{z}^{iT}(\xi) - \eta \overline{q}^T(\xi, j_\xi) > r(\xi) \cdot \overline{z}^{iT}(\xi^-).$$

Hence, $1 + \overline{p}^T(\xi) \cdot (\overline{x}^{iT}(\xi) - \omega^i(\xi)) + \overline{q}^T(\xi) \cdot \overline{z}^{iT}(\xi) > r(\xi) \overline{z}^{iT}(\xi^-) + \eta \overline{q}^T(\xi, j_\xi) +$ $(1-\beta) \overline{q}^T(\xi) \cdot \overline{z}^{iT}(\xi)$.

Since $\overline{p}^T(\xi) \cdot (\overline{x}^{iT}(\xi) - \omega^i(\xi)) + \overline{q}^T(\xi) \cdot \overline{z}^{iT}(\xi) - r(\xi) \overline{z}^{iT}(\xi^-) \leqslant 0$, it follows that one must have $1 \geqslant 1 - (1-\beta)\, \overline{q}^T(\xi) \cdot \overline{z}^{iT}(\xi) > \eta \overline{q}^T(\xi, j_\xi)$. Hence,

$$\text{(4.1)} \qquad \overline{q}^T(\xi, j_\xi) < \frac{\| e \|_1}{(1-\beta) M'}$$

By Condition (iii) of Lemma 4.1, one gets
$\forall j \in J(\xi), \overline{\pi}^i_T(\xi)\, \overline{q}^T(\xi, j) = \sum_{\xi' \in \xi^+} \overline{\pi}^i_T(\xi')\, r^j(\xi')$.

Then, $\forall j \in J(\xi),\ |\overline{q}^T(\xi, j)| \leqslant \sum_{\xi' \in \xi^+} \frac{\overline{\pi}^i_T(\xi')}{\overline{\pi}^i_T(\xi)} |r^j(\xi')|$ and by the inequality (4.1) above, one gets that $\overline{q}^T(\xi, j_\xi) = \sum_{\xi' \in \xi^+} \frac{\overline{\pi}^i_T(\xi')}{\overline{\pi}^i_T(\xi)} < \frac{\|e\|_1}{(1-\beta) M'}$. Consequently, $\forall j \in J(\xi)$, one gets $|\overline{q}^T(\xi, j)| \leqslant \frac{\|e\|_1}{(1-\beta) M'} \gamma(\xi)$, where $\gamma(\xi) = \max_{\substack{j \in J(\xi) \\ \xi' \in \xi^+}} r^j(\xi')$.

– *Bounds on* $\overline{q}^T(\xi) \cdot \overline{z}^{iT}(\xi)$:

- Let $i \in I_1 \cap I(\xi)$. As in SEGHIR [25], one gets the following bounds:

$$-\frac{\operatorname{card} I(\xi) - 1}{1 - \beta^i} \leqslant \overline{q}^T(\xi) \cdot \overline{z}^{iT}(\xi) \leqslant \frac{1}{1 - \beta^i}.$$

- Let $i \in I_2 \cap I(\xi)$. Using the same trick that in SEGHIR [25], the last bounds on $\overline{q}^T(\xi) \cdot \overline{z}^{iT}(\xi)$ for the finite lived agents and Assumption (A.9), one gets bounds on $\overline{q}^T(\xi) \cdot \overline{z}^{iT}(\xi)$ for the infinite lived agents.

– *Bounds on* $\overline{z}^{iT}(\xi)$:

Recall that:

$$[\overline{p}^T(\xi')\cdot(\overline{x}^{iT}(\xi')-\omega^i(\xi'))+\overline{q}^T(\xi')\cdot\overline{z}^{iT}(\xi')]_{\xi'\in\xi^+}=R(\xi)\overline{z}^{iT}(\xi).$$

Since the left term of the last equality is bounded, in view of Assumption (A.8), $\overline{z}^{iT}(\xi)$ is bounded (to see this, one can use for example the Cramer's formula to compute $\overline{z}^{iT}(\xi)$).

4.3 Implicit Debt Systems

This step is devoted to the construction of suitable debt systems as limits of implicit debt systems of the economies $\mathcal{E}^T$. Let ξ be a node such that $t(\xi)\leqslant T$ and let i be an infinite lived agent (i.e.: $i\in I_2$). I associate, to the equilibrium of each truncated economy $\mathcal{E}^T$, an implicit system of debt constraints. I define the implicit debt system $\overline{D}^{iT}(\xi)$ at node ξ as follows:

$$\overline{D}^{iT}(\xi)=\inf\{\overline{p}^T(\xi)\cdot((x^i(\xi)-\omega^i(\xi))+\overline{q}^T(\xi)\cdot z^i(\xi)\},$$

where the infimum is taken among all (x^i, z^i) satisfying the spot budget constraints at all nodes of $D(\xi)\cap D^T$ and such that for each node σ such that $t(\sigma)=T$, $z^i(\sigma)=0$. Let us remark that $\forall\xi: t(\xi)\leqslant T,\ \overline{D}^{iT}(\xi)\leqslant 0$.

Note that, by definition of $\overline{D}^{iT}$, one has for each agent $i\in I_2$:

$$\left\{(x^i,z^i)\ \middle|\ \begin{array}{l}\forall\xi\ :\ t(\xi)\geqslant T,\ z^i(\xi)=0,\\ \forall\xi\ :\ t(\xi)>T,\ x^i(\xi)=0,\\ \forall\xi\in D^T,\\ \overline{p}^T(\xi)\cdot(x^i(\xi)-\omega^i(\xi))+\overline{q}^T(\xi)\cdot z^i(\xi)\leqslant r(\xi)\cdot z^i(\xi^-).\end{array}\right\}.$$

$$=\left\{(x^i,z^i)\ \middle|\ \begin{array}{l}\forall\xi\in D^T, r(\xi)\cdot z^i(\xi^-)\geqslant\overline{D}^{iT}(\xi)\\ \forall\xi\ :\ t(\xi)\geqslant T,\ z^i(\xi)=0,\\ \forall\xi\ :\ t(\xi)>T,\ x^i(\xi)=0,\\ \forall\xi\in D^T,\\ \overline{p}^T(\xi)\cdot(x^i(\xi)-\omega^i(\xi))+\overline{q}^T(\xi)\cdot z^i(\xi)\leqslant r(\xi)\cdot z^i(\xi^-).\end{array}\right\}.$$

LEMMA 4.3: *For each infinitely lived agent* $i \in I_2$, $\overline{D}^{iT}$ *is* $(\overline{p}^T, \overline{q}^T)$ *– loose and consistent at each node in* D^T.

See Appendix A for a proof.

LEMMA 4.4: $\forall i \in I_2$, $\forall \xi \,:\, t(\xi) \leqslant T$, $\overline{D}^{iT}(\xi)$ *is bounded.*

See Appendix A for a proof.

5 Asymptotic Results

In this section, we show that the equilibria of the truncated (finite horizon) economies has convergent subsequences and that the cluster point is an equilibrium with debt constraints for the original (infinite horizon) economy.

5.1 Convergence

Let $ba(D \times G)$ be the norm dual of $\ell_\infty(D \times G)$ consisting of bounded finitely additive set functions on $D \times G$ and let $\|\cdot\|_{ba}$ be the norm of $ba(D \times G)$. The prices $(\overline{P}^i_T,\ T \in \mathcal{T})$ can be viewed as elements of $ba(D \times G)$. Let $\sigma(ba, \ell_\infty)$ be the weak* topology of ba. I recall (see BEWLEY [6] for a proof) that the set $A := \{y \in ba(D \times G) \mid y \geqslant 0,\ \|y\|_{ba} = 1\}$ is compact for $\sigma(ba, \ell_\infty)$. In view of condition (iv) of Lemma 4.1, one deduces that $\overline{P}^i_T \cdot 1 = \|\overline{P}^i_T\|_{ba} = 1$, and $\overline{P}^i_T \in A$, $\forall i \in I^T$, $\forall T \in \mathcal{T}$. Then, there exists a directed set $(\Lambda, \geqslant)$ and a subnet $((\overline{P}^i_{T_\lambda},\ i \in I^T),\ \lambda \in (\Lambda, \geqslant))$ such that for each $i \in I$, $\overline{P}^i_{T_\lambda}$ converges to some $\overline{P}^i$ for the topology $\sigma(ba, \ell_\infty)$. Moreover, by Tychonov's Theorem, one has the existence of a directed set $(\Sigma, \geqslant)$ and a subnet $((\overline{x}^{iT_\sigma}, \overline{z}^{iT_\sigma}, \overline{D}^{iT_\sigma}, \overline{\pi}^{iT_\sigma})_{i \in I^T}, \overline{p}^{T_\sigma}, \overline{q}^{T_\sigma})$ which converges to some $((\overline{x}^i, \overline{z}^i, \overline{\pi}^i)_{i \in I}, (\overline{D}^i)_{i \in I_2}, \overline{p}, \overline{q})$ for the product topology. Moreover, one has at the limit: $\forall \xi \in D$, $\overline{p}(\xi) \cdot (\overline{x}^i(\xi) - \omega^i(\xi)) + \overline{q}(\xi) \cdot \overline{z}^i(\xi) \leqslant r(\xi) \cdot z^i(\xi^-)$, $\sum_{i \in I(\xi)} (\overline{x}^i(\xi) - \omega^i(\xi)) = 0$ and $\sum_{i \in I(\xi)} \overline{z}^i(\xi) = 0$. One has also that for each node $\xi \in D$, $\overline{p}(\xi) \cdot e = 1$.

5.2 Looseness and Consistency of the Debt System $\overline{D}^i$

LEMMA 5.1: $\forall i \in I_2$, $\overline{D}^i$ *is* $(\overline{p}, \overline{q})$ *– loose and consistent.*

See Appendix A for a proof.

5.3 **Optimality of** $(\overline{x}^i, \overline{z}^i)$

LEMMA 5.2: *For each finitely lived agent* $i \in I_1$, $(\overline{x}^i, \overline{z}^i)$ *is optimal in* $B^i(\overline{p}, \overline{q})$ *and for each infinitely lived agent* $i \in I_2$, $(\overline{x}^i, \overline{z}^i)$ *is optimal in* $B^i(\overline{p}, \overline{q}, \overline{D}^i)$.

Proof: For finitely lived agents, the optimality was proved in SEGHIR [25]. In this paper, I will only prove the optimality of the infinite lived agents i in I_2, i.e.: the optimality of $(\overline{x}^i, \overline{z}^i)$ in $B^i(\overline{p}, \overline{q}, \overline{D}^i)$. By contradiction, let us suppose that there exists $i \in I_2$ and there exists (x^i, z^i) in $B^i(\overline{p}, \overline{q}, \overline{D}^i)$ such that $x^i \succ^i \overline{x}^i$. Note that since $\sigma(\ell_\infty, \ell_1)$ and $\tau(\ell_\infty, \ell_1)$ coincide on bounded subsets of ℓ_∞, one has that $\overline{x}^{iT} \to \overline{x}^i$ also for $\tau(\ell_\infty, \ell_1)$. In view of Corollary 15.29 (p. 505), in ALIPRANTIS–BORDER [1], it follows from the Mackey continuity of $\succeq^i$ that there exists an integer N and there exists $\overline{\theta}$ such that $\forall \theta \geqslant \overline{\theta}$, one has $x^i \chi_{D^N} \succ^i \overline{x}^{iT^{\theta}}$. For $\epsilon > 0$, let us define $\underline{x}^i = (1-\epsilon)\, x^i \chi_{D^N}$ and $\underline{z}^i = (1-\epsilon)\, z^i \chi_{D^N}$. Then for each node $\xi : t(\xi) \leqslant N$, one has:

$$\begin{aligned}
\overline{p}(\xi)\cdot(\underline{x}^i(\xi)-\omega^i(\xi))+\overline{q}(\xi)\cdot\underline{z}^i(\xi) &= \overline{p}(\xi)\cdot[(1-\epsilon)\,x^i(\xi)-\omega^i(\xi)]+\overline{q}(\xi)\cdot z^i(\xi)\\
&\leqslant (1-\epsilon)\,[\overline{p}(\xi)\cdot(x^i(\xi)-\omega^i(\xi))+\overline{q}(\xi)\cdot z^i(\xi)]\\
&\leqslant (1-\epsilon)\,r(\xi)\cdot z^i(\xi^-)\\
&\leqslant r(\xi)\cdot\underline{z}^i(\xi^-).
\end{aligned}$$

Since $\overline{D}^i$ is $(\overline{p}, \overline{q})-$loose, one has $\overline{D}^i(\xi) \leqslant -\overline{p}(\xi)\cdot\omega^i(\xi))$, $\forall \xi \in D$. Moreover, since $\overline{p}(\xi) \geqslant 0$, and different from 0, and $\omega^i(\xi) \gg 0$, $\forall \xi \in D$, one gets $\overline{D}^i(\xi) < 0$, $\forall \xi \in D$. Hence, $\forall \xi :\ t(\xi) \leqslant N+1$, one has $r(\xi)\cdot\underline{z}^i(\xi^-) > \overline{D}^i(\xi)$. Then, $(\underline{x}^i, \underline{z}^i)$ satisfies strictly the budget constraints at each node $\xi : t(\xi) \leqslant N$ and the debt constraints at each node $\xi : t(\xi) \leqslant N+1$. Since there is only a finite number of strict inequalities, one gets that there exists $\overline{T} > N$ such that $\forall\, T > \overline{T}$, one has that $(\underline{x}^i, \underline{z}^i)$ satisfies strictly the budget constraints, with $(\overline{p}^T, \overline{q}^T)$, at each node $\xi : t(\xi) \leqslant N$ and the implicit debt constraints, $\overline{D}^{iT}$, at each node $\xi : t(\xi) \leqslant N+1$. Without loss of generality, one can assume that $\overline{T} > T^{\overline{\theta}}$. Since $\overline{D}^{iT}$ is $(\overline{p}^T, \overline{q}^T)-$consistent, one can define some portfolio $\tilde{z}^{iT}$ which is equal to 0 from period T, $\tilde{z}^{iT}$ coincides with $(1-\epsilon)z^i$ for the nodes $\xi : t(\xi) \leqslant N$ and such that $(\underline{x}^i, \tilde{z}^{iT}) \in B^{iT}(\overline{p}^T, \overline{q}^T)$. For T sufficiently large and ϵ sufficiently small, by the Mackey continuity, one has $\underline{x}^i \succ^i \overline{x}^{iT}$, which contradicts the optimality of $(\overline{x}^{iT}, \overline{z}^{iT})$ in $B^{iT}(\overline{p}^T, \overline{q}^T)$. Consequently, for each infinitely lived agent i in I_2, $(\overline{x}^i, \overline{z}^i)$ is optimal in $B^{iT}(\overline{p}, \overline{q})$.

6 Concluding Remarks

In this paper, using systems of debt constraints, due to LEVINE–ZAME [18], I showed the equilibrium existence in an infinite horizon incomplete market model with one-period numeraire assets, a demographic structure in which finitely-lived and infinitely-lived agents coexist and a stochastic structure characterized by a finite number of successors at each node of the event-tree. As mentioned in the introduction of the paper, one could prove the existence of equilibrium in such a model using a transversality condition but the proof of MAGILL–QUINZII [19] has to be remade to hold in our model. Moreover, our proof can be easily adapted to prove the equilibrium existence in the nominal case.

On the other hand, the numeraire assets considered in this paper are one-period of maturity. One could also study a model with infinitely-lived assets to emphasize the phenomena of bubbles (the excess of the price of an asset over its fundamental value) studied by MAGILL–QUINZII [20] and by SANTOS–WOODFORD [23].

As the most overlapping generation models (see for instance SCHMACHTENBERG [24], FLORENZANO–GOURDEL–PÁSCOA [12], SEGHIR [25]...), we did not allow default in our model; that is the borrowers pay back fully their debts. It is also interesting to consider a model in which default is allowed and to require the borrowers to constitute collateral, in term of durable goods, which will be seized and given to the lenders in case of default (see GEANAKOPLOS–ZAME [13] for collateral requirement). We can also introduce some utility penalties in case of default (see DUBEY–GEANAKOPLOS–SHUBIK [9] for details on the utility penalties). This will permit: (i) to prove the existence of equilibrium with real assets, (ii) to allow the finitely-lived agents to attend the financial markets even at the last period of their lifetime duration (since at this period their promise is secured by the collateral that they have to constitute) and (iii) to introduce a structure of heritage (since durable goods will be considered and an heritage structure will avoid the loss of the depreciated commodities of an agent after his death). Moreover, a structure of altruism can be considered in such a model. ■

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Appendix A

PROOF OF LEMMA 4.2: Let $\xi \in D$ and $i \in I_2$. By Assumption (A.7), one has $x^i + e^{\xi} \succ^i x^i$, $\forall x^i \in \hat{X}^i$. The Mackey continuity of $\succeq^i$ implies that for each $x^i \in \hat{X}^i$, there exists a neighborhood V_{x^i} of x^i and there exists $\alpha^i_\xi(x^i) \in]0,1[$ such that:

$$\alpha^i_\xi(x^i)\, x'^i + e^{\xi} \succ^i x'^i, \ \forall x'^i \in V_{x^i}.$$

Since $\hat{X}^i$ is $\sigma(\ell_\infty(D \times G), \ell_1(D \times G))$–compact and since, on bounded subsets of $\ell_\infty(D \times G)$, the topologies $\sigma(\ell_\infty(D \times G), \ell_1(D \times G))$ and $\tau(\ell_\infty(D \times G), \ell_1(D \times G))$ coincide[5] on $\hat{X}^i$, one gets that $\hat{X}^i$ is $\tau(\ell_\infty(D \times G), \ell_1(D \times G))$–compact. Hence, one can extract a finite subcovering $V_{x^i_1}, \ldots, V_{x^i_p}$ of $\hat{X}^i$ and consequently α^i_ξ can be chosen uniform on $x^i \in \hat{X}^i$.

The existence of α^i_ξ for finitely lived agents was proved in SEGHIR [25].

The conclusion of Lemma 4.2. follows from the condition (i) of Assumption (A.3) and the monotonicity of preferences.

PROOF OF LEMMA 4.3: Some adaptations of the definition of $(\overline{p}^T, \overline{q}^T)$ – looseness and consistency at terminal nodes of $\mathcal{E}^T$ (i.e.: at nodes $\xi : t(\xi) = T$) are required. A system of debts D^i defined on D^T is $(\overline{p}^T, \overline{q}^T)$ – loose at a node $\xi : t(\xi) = T$ if $D^i(\xi) + \overline{p}^T(\xi) \cdot \omega^i(\xi) \leqslant 0$. The system D^i is $(\overline{p}^T, \overline{q}^T)$ – consistent at a node $\xi : t(\xi) = T$ if $D^i(\xi) + \overline{p}^T(\xi) \cdot \omega^i(\xi) \leqslant 0$.

Hence, a system of debt constraints D^i is $(\overline{p}^T, \overline{q}^T)$ – loose and consistent at a node $\xi : t(\xi) = T$ if and only if $D^i(\xi) = -\overline{p}^T(\xi) \cdot \omega^i(\xi)$. Then, by definition of the implicit system of debts, $\overline{D}^{iT}$ is $(\overline{p}^T, \overline{q}^T)$ – loose and consistent at each node $\xi : t(\xi) = T$.

Let ξ be a node such that $t(\xi) = t < T$ and let $z^i(\xi^-) \in \mathbf{R}^{J(\xi^-)}$ be a portfolio such that $r(\xi) \cdot z^i(\xi^-) \geqslant \overline{D}^{iT}(\xi)$. It follows from the non-emptiness of the budget sets of i that there exists $(x^i(\xi), z^i(\xi))$ such that $r(\xi) \cdot z^i(\xi^-) + \overline{p}^T(\xi) \cdot (\omega^i(\xi) - x^i(\xi)) - \overline{q}^T(\xi) \cdot z^i(\xi) \geqslant 0$. Then, $r(\xi) \cdot z^i(\xi^-) + \overline{p}^T(\xi) \cdot \omega^i(\xi) - \overline{q}^T(\xi) \cdot z^i(\xi) \geqslant 0$. Moreover, by definition of $\overline{D}^{iT}(\xi')$ and in view of the budget constraints at each $\xi' \in \xi^+$, one gets that $\overline{D}^{iT}(\xi') \leqslant r(\xi') \cdot z^i(\xi)$, $\forall \xi' \in \xi^+$.

Then, $\overline{D}^{iT}$ is $(\overline{p}^T, \overline{q}^T)$ – consistent at $\xi : t(\xi) < T$. Consequently, $\overline{D}^{iT}$ is $(\overline{p}^T, \overline{q}^T)$ – consistent at each node $\xi \in D^T$.

5. See Appendix B for a proof.

Let us show that $\bar{D}^{iT}$ is $(\overline{p}^T, \overline{q}^T)$ – loose at each $\xi : t(\xi) \leqslant T - 1$. To this end, let us consider a node ξ such that $t(\xi) = t < T$ and let $z^i(\xi) \in \mathbf{R}^{J(\xi)}$ be a portfolio such that $r(\xi') \cdot z^i(\xi) \geqslant \bar{D}^{iT}(\xi')$, $\xi' \in \xi^+$. By definition of the implicit debt system $\bar{D}^{iT}$, one has $\bar{D}^{iT}(\xi) \leqslant -\overline{p}^T(\xi) \cdot \omega^i(\xi) + \overline{q}^T(\xi) \cdot z^i(\xi)$. Then, $\bar{D}^{iT}(\xi) + \overline{p}^T(\xi) \cdot \omega^i(\xi) - \overline{q}^T(\xi) \cdot z^i(\xi) \leqslant 0$. Hence, $\bar{D}^{iT}$ is $(\overline{p}^T, \overline{q}^T)$ – loose at the node ξ. Consequently, $\bar{D}^{iT}$ is $(\overline{p}^T, \overline{q}^T)$ – loose at each node $\xi \in D^T$.

Proof of Lemma 4.4: Let i be an infinite lived agent (i.e.: $i \in I_2$).

I recall that, by definition of the implicit debt system, one has $\bar{D}^{iT}(\xi) \leqslant 0, \forall \xi : t(\xi) \leqslant T$. Let us show that $\forall \xi : t(\xi) \leqslant T$, $\bar{D}^{iT}(\xi)$ is bounded from below. By contradiction, suppose that there exists $\xi : t(\xi) \leqslant T$ such that $\bar{D}^{iT}(\xi)$ is not bounded from below. Since $\bar{D}^{iT}$ is $(\overline{p}^T, \overline{q}^T)$ – consistent one has, for each node $\sigma \in D^T$, there exists a portfolio $z^i(\sigma) \in \mathbf{R}^{J(\sigma)}$ such that:

$$\begin{cases} \bar{D}^{iT}(\sigma) + \overline{p}^T(\sigma) \cdot \omega^i(\sigma) - \overline{q}^T(\sigma) \cdot z^i(\sigma) \geqslant 0, \\ \quad \text{and } r(\sigma') \cdot z^i(\sigma) \geqslant \bar{D}^{iT}(\sigma), \qquad \forall \sigma' \in \sigma^+. \end{cases}$$

Let us consider the following change on the consumption-portfolio plan of this agent i from node ξ onwards:

$$\tilde{x}^i = \begin{cases} \overline{x}^{iT}(\xi) + e \\ \beta \overline{x}^{iT}(\xi'), \quad \forall \xi' \in D^+(\xi) \end{cases}$$

$$\tilde{z}^i(\sigma) = \beta z^i(\sigma) - \delta\,(0, \ldots, 0, 1, 0, \ldots, 0), \forall \sigma \geqslant \xi,$$

where $\delta < 0$ and the component 1 in the vector (0,..., 0, 1, 0,..., 0) denotes one unit of the non-risky asset j_σ.

At each $\xi' \in \xi^+$, $(\tilde{x}^i, \tilde{z}^i)$ satisfies the budget constraint if: $\overline{p}^T(\xi') \cdot (\beta \overline{x}^{iT}(\xi') - \omega^i(\xi')) + \beta \overline{q}^T(\xi') \cdot \overline{z}^{iT}(\xi') - \delta \overline{q}^T(\xi', j_{\xi'}) \leqslant r(\xi') \cdot [\beta z^i(\xi) - \delta(0, \ldots, 0, 1, 0, \ldots, 0)]$.

Since $\overline{p}^T(\xi') \cdot (\beta \overline{x}^{iT}(\xi') - \omega^i(\xi')) + \beta \overline{q}^T(\xi') \cdot \overline{z}^{iT}(\xi') \leqslant \beta\,[\overline{p}^T(\xi') \cdot (\overline{x}^{iT}(\xi') - \omega^i(\xi')) + \overline{q}^T(\xi') \cdot \overline{z}^{iT}(\xi')]$, one gets $\overline{p}^T(\xi') \cdot (\beta \overline{x}^{iT}(\xi') - \omega^i(\xi')) + \beta \overline{q}^T(\xi') \cdot \overline{z}^{iT}(\xi') - \delta \overline{q}^T(\xi', j_{\xi'}) \leqslant \beta\,[\overline{p}^T(\xi') \cdot \overline{x}^{iT}(\xi') + \bar{D}^{iT}(\xi')] - \delta \overline{q}^T(\xi', j_{\xi'})$. Recalling that for each $\xi' \in \xi^+$, one has by (A.10) that $r^{j_\xi}(\xi') = 1$, we deduce that $(\tilde{x}^i, \tilde{z}^i)$ satisfies the budget constraint at $\xi' \in \xi^+$, if:

$$\delta(1 - \overline{q}^T(\xi', j_{\xi'})) \leqslant \beta\,[r(\xi') \cdot z^i(\xi) - \bar{D}^{iT}(\xi')] + \beta \overline{p}^T(\xi') \cdot \overline{x}^{iT}(\xi').$$

Since $r(\xi')\cdot z^i(\xi)-\overline{D}^{iT}(\xi')\geqslant 0$, one has $(\tilde{x}^i,\tilde{z}^i)$ satisfies the budget constraint at $\xi'\in\xi^+$ if: $\delta(1-\overline{q}^T(\xi',j_{\xi'}))\leqslant\beta\overline{p}^T(\xi')\cdot\overline{x}^{iT}(\xi')$. Recalling that $\overline{q}^T(\xi',j_{\xi'})\leqslant\frac{\|e\|_1}{(1-\beta)M'}$ and that $\delta<0$, one has that $(\tilde{x}^i,\tilde{z}^i)$ satisfies the budget constraint at $\xi'\in\xi^+$, if:

$$\delta\left(1-\frac{\|e\|_1}{(1-\beta)M'}\right)\leqslant\beta\overline{p}^T(\xi')\cdot\overline{x}^{iT}(\xi'). \tag{6.1}$$

Since $\forall\xi'\in\xi^+$, $0\leqslant\overline{p}^T(\xi')\cdot\overline{x}^{iT}(\xi')\leqslant\frac{GM}{(1-\alpha_\xi)M'}$, one can choose δ satisfying (6.1) and consequently $(\tilde{x}^i,\tilde{z}^i)$ will satisfy the budget constraints at each node $\xi'\in\xi^+$. At node ξ, $(\tilde{x}^i,\tilde{z}^i)$ satisfies the budget constraint if:

$$\overline{p}^T(\xi)\cdot(\overline{x}^{iT}(\xi)-\omega^i(\xi))+1+\beta\,\overline{q}^T(\xi)\cdot z^i(\xi)-\delta\,\overline{q}^T(\xi,j_\xi)\leq r(\xi)\cdot\overline{z}^{iT}(\xi^-).$$

Since $\overline{p}^T(\xi)\cdot(\overline{x}^{iT}(\xi)-\omega^i(\xi))+1+\beta\,\overline{q}^T(\xi)\cdot z^i(\xi)-\delta\,\overline{q}^T(\xi,j_\xi)\leqslant\overline{p}^T(\xi)\cdot\overline{x}^{iT}(\xi)+1+\beta\overline{D}^{iT}(\xi)-\delta\,\overline{q}^T(\xi,j_\xi)$, one has $(\tilde{x}^i,\tilde{z}^i)$ satisfies the budget constraint at ξ if: $\beta\overline{D}^{iT}(\xi)\leqslant r(\xi)\cdot\overline{z}^{iT}(\xi^-)-\overline{p}^T(\xi)\cdot\overline{x}^{iT}(\xi)+\delta\,\overline{q}^T(\xi,j_\xi)-1$. Since $\overline{D}^{iT}(\xi)$ is unbounded from below then for T sufficiently large, this last inequality can be satisfied and hence $(\tilde{x}^i,\tilde{z}^i)$ satisfies the budget constraint at ξ. In view of Assumption (A.9), this contradicts the optimality of $(\overline{x}^{iT},\overline{z}^{iT})$. Consequently, for each node ξ such that $t(\xi)\leqslant T$, $\overline{D}^{iT}(\xi)$ is bounded.

Proof of Lemma 5.1: To show that $\overline{D}^i$ is $(\overline{p},\overline{q})$ – loose at a node ξ in D, let us consider a portfolio $z^i(\xi)\in\mathbf{R}^{J(\xi)}$ such that for each $\xi'\in\xi^+$, $r(\xi')\cdot z^i(\xi)\geqslant\overline{D}^i(\xi')$. I want to prove that $\overline{D}^i(\xi)+\overline{p}(\xi)\cdot\omega^i(\xi)-\overline{q}(\xi)\cdot z^i(\xi)\leqslant 0$. For $\epsilon>0$, let us define a portfolio $\tilde{z}^i(\xi)$ by: $\tilde{z}^i(\xi):=z^i(\xi)+\epsilon\gamma(\xi)$, where $\gamma(\xi)=(0,...,0,1,0,...,0)$ and the component 1 in the vector (0,..., 0, 1, 0,..., 0) denotes one unit of the non-risky asset j_ξ. For $\epsilon>0$, one has for each node $\xi'\in\xi^+$, $r(\xi')\cdot\tilde{z}^i(\xi)>r(\xi')\cdot z^i(\xi)\geqslant\overline{D}^i(\xi')$. Hence, there exists $\overline{\theta}$ such that $\forall\theta\geqslant\overline{\theta}$, one has $r(\xi')\cdot\tilde{z}^i(\xi)\geqslant\overline{D}^{iT^0}(\xi')$. Since $\overline{D}^{iT^0}$ is $(\overline{p}^{T^0},\overline{q}^{T^0})$ – loose, one gets $\overline{D}^{iT^0}(\xi)+\overline{p}^{T^0}(\xi)\cdot\omega^i(\xi)-\overline{q}^{T^0}(\xi)\cdot z^i(\xi)\leqslant 0$. Moreover, since $\overline{D}^{iT^0}\to\overline{D}^i$, $\overline{p}^{T^0}\to\overline{p}$ and $\overline{q}^{T^0}\to\overline{q}$, one gets, when θ goes to infinity, that $\overline{D}^i(\xi)+\overline{p}(\xi)\cdot\omega^i(\xi)-\overline{q}(\xi)\cdot z^i(\xi)\leqslant 0$. Hence, $\overline{D}^i$ is $(\overline{p},\overline{q})$ – loose at each node ξ in D.

To show that $\overline{D}^i$ is $(\overline{p},\overline{q})$ – consistent at a node ξ in D, let us consider a portfolio $z^i(\xi^-)\in\mathbf{R}^{J(\xi^-)}$ such that $r(\xi)\cdot z^i(\xi^-)\geqslant\overline{D}^i(\xi)$. For $\epsilon>0$, let us define a portfolio $z^i(\xi^-)\in\mathbf{R}^{J(\xi^-)}$ by: $\tilde{z}^i(\xi^-):=z^i(\xi^-)+\epsilon\gamma(\xi^-)$, where $\gamma(\xi^-)=(0,...,0,1,0,...,0)$ and the component 1 in the vector (0,..., 0, 1, 0,..., 0) denotes one unit of the non-risky asset j_{ξ^-}. For $\epsilon>0$, one has $r(\xi)\cdot\tilde{z}^i(\xi^-)>r(\xi)\cdot z^i(\xi)\geqslant\overline{D}^i(\xi)$.

Hence, there exists $\overline{\theta}$ such that $\forall\theta \geqslant \overline{\theta}$, one has $r(\xi)\cdot\tilde{z}^i(\xi^-) \geqslant \overline{D}^{iT^0}(\xi)$. Since $\overline{D}^{iT^0}$ is $(\overline{p}^{T^0}, \overline{q}^{T^0})-$consistent, it follows from Remark 3.3 that, there exists $\tilde{z}^{iT^0}(\xi) \in \mathbf{R}^{J(\xi)}$ such that:

$$(6.2) \qquad r(\xi)\cdot\tilde{z}^i(\xi^-) + \overline{p}^{T^0}(\xi)\cdot\omega^i(\xi) - \overline{q}^{T^0}(\xi)\cdot\tilde{z}^{iT^0}(\xi) \geqslant 0,$$

$$(6.3) \qquad \text{and } r(\xi')\cdot\tilde{z}^{iT^0}(\xi) \geqslant \overline{D}^{iT^0}(\xi'),\ \forall\xi' \in \xi^+.$$

Since $\overline{p}^{T^0}(\xi)$ is bounded, it follows from (6.2) that $\overline{q}^{T^0}(\xi)\cdot\tilde{z}^{iT^0}(\xi)$ is bounded from above. Moreover, since $\overline{D}^{iT^0}(\xi')$ is bounded then from (6.3), one gets that $\tilde{z}^{iT^0}(\xi)$ is bounded from below. Let us show that $\tilde{z}^{iT^0}(\xi)$ is bounded from above. By contradiction, let us suppose that $\tilde{z}^{iT^0}(\xi) \to +\infty$. Then, for each $\xi' \in \xi^+$, $r(\xi')\cdot\tilde{z}^{iT^0}(\xi) \to +\infty$. For each $T^\theta > t(\xi)$, let us define a portfolio $\tilde{z}^{iT^0}$ by:

$$\tilde{z}^{iT^\theta} = \begin{cases} \overline{z}^{iT^\theta}(\xi') & \forall\xi' \notin D(\xi) \\ z^{iT^\theta}(\xi) - \delta(0,\dots,0,1,0,\dots,0) & \text{at node } \xi \\ \beta\overline{z}^{iT^0}(\xi') & \forall\xi' \in D^+(\xi). \end{cases}$$

with $\delta > 0$ and the component 1 in the vector (0,..., 0, 1, 0,..., 0) denotes one unit of the Non-risky asset j_ξ.
For T^θ sufficiently large, one can choose $\delta > 0$ such that:

$$(\overline{x}^{iT^0}\chi_{D\setminus D^+(\xi)} + \beta\overline{x}^{iT^0}\chi_{D^+(\xi)} + e^\xi,\ \tilde{z}^{iT^0}) \in B^{iT^0}(\overline{p}^{T^0}, \overline{q}^{T^0}).$$

Indeed, for each node $\xi' \in \xi^+$, one can choose, for T^θ sufficiently large, $\delta > 0$ satisfying $\overline{p}^{T^0}(\xi')\cdot(\beta x^{iT^0}(\xi') - \omega^i(\xi')) + \beta\overline{q}^{T^0}(\xi')\cdot\tilde{z}^{iT^0}(\xi') \leqslant r(\xi')\cdot z^{iT^0}(\xi) - \delta$.
At node ξ, to satisfy the budget constraints, one must have:

$$(6.4)\ \overline{p}^{T^0}(\xi)\cdot(x^{iT^0}(\xi) - \omega^i(\xi)) + 1 + \overline{q}^{T^0}(\xi)\cdot z^{iT^0}(\xi) - \delta\overline{q}^{T^0}(\xi, j_\xi) \leqslant r(\xi)\cdot\overline{z}^{iT^0}(\xi^-).$$

This last inequality is satisfied if

$$\overline{p}^{T^0}(\xi)\cdot(x^{iT^0}(\xi) - \omega^i(\xi)) + \overline{q}^{T^0}(\xi)\cdot z^{iT^0}(\xi) - \delta\overline{q}^{T^0}(\xi, j_\xi) \leqslant r(\xi)\cdot\overline{z}^{iT^\theta}(\xi^-).$$

Moreover, since $(\overline{x}^{iT^0}, \overline{z}^{iT^0})$ satisfies the budget constraint at node ξ, the last inequality is satisfied if $1+\overline{q}^{T^0}(\xi)\cdot z^{iT^0}(\xi)-\delta\overline{q}^{T^0}(\xi, j_\xi)\leqslant \overline{q}^{T^0}(\xi)\cdot \overline{z}^{iT^0}(\xi)$. Then, the inequality (6.4) is satisfied if:

$$\text{(6.5)} \qquad \delta\overline{q}^{T^0}(\xi, j_\xi)\geqslant 1+\overline{q}^{T^0}(\xi)\cdot z^{iT^0}(\xi)-\overline{q}^{T^0}(\xi)\cdot \overline{z}^{iT^0}(\xi).$$

Since the right term of the last inequality is bounded from below, one can choose δ satisfying (6.5). Hence, for T^θ sufficiently large, one can choose some real $\delta > 0$ such that $(\overline{x}^{iT^0}\chi_{D\setminus D^+(\xi)}+\beta\overline{x}^{iT^0}\chi_{D^+(\xi)}+e^\xi,\ \tilde{z}^{iT^0})\in B^{iT^0}(\overline{p}^{T^0}, \overline{q}^{T^0})$. Since, by Assumption (A.9), $\overline{x}^{iT^0}\chi_{D\setminus D^+(\xi)}+\beta\overline{x}^{iT^0}\chi_{D^+(\xi)}+e^\xi \succ {}^i\overline{x}^{iT^0}$, it contradicts the optimality of $(\overline{x}^{iT^0}, \overline{z}^{iT^0})$. Hence, $\tilde{z}^{iT^0}(\xi)$ is bounded. Then, $\tilde{z}^{iT^0}(\xi)$ converges to some $z^i(\xi)\in \mathbf{R}^{J(\xi)}$ and one has, when θ goes to infinity,

$$\begin{cases} r(\xi)\cdot\tilde{z}^i(\xi^-)+\overline{p}(\xi)\cdot\omega^i(\xi)-\overline{q}(\xi)\cdot\tilde{z}^i(\xi)\geqslant 0 \\ \qquad \text{and } r(\xi')\cdot\tilde{z}^i(\xi)\geqslant \overline{D}^i(\xi'), \qquad \forall \xi'\in\xi^+. \end{cases}$$

Then $\overline{D}^i$ is $(\overline{p},\overline{q})-$consistent at each node ξ in D.

Appendix B

The purpose of this part is to prove that the topologies $\sigma(\ell_\infty, \ell_1)$ and $\tau(\ell_\infty, \ell_1)$ coincide on bounded subsets of ℓ_∞. Let us define the following sets:

$$\overline{B_\infty} = \{f \in \ell_\infty \mid \| f \|_\infty \leqslant 1\} \text{ and } \overline{B_1} = \{x \in \ell_1 \mid \| x \|_1 \leqslant 1\}.$$

Recall that $\sigma(\ell_\infty, \ell_1) \subset \tau(\ell_\infty, \ell_1)$ (i.e.: $\tau(\ell_\infty, \ell_1)$ is finer than $\sigma(\ell_\infty, \ell_1)$).

Let B be a bounded subset of ℓ_∞. Then, there exists $\alpha > 0$ such that $B \subset \alpha\overline{B_\infty}$. Recall also that a neighborhood base of 0 for:

(i): $\tau(\ell_\infty, \ell_1)$ is composed by the sets of the form:

$V_K^\epsilon := \{x \in B \mid |f(x)| < \epsilon, \ \forall f \in K\}$, $\epsilon > 0$ and K a convex, $\sigma(\ell_\infty, \ell_1)$–compact and circled subset of ℓ_1,

(ii): $\sigma(\ell_\infty, \ell_1)$ is composed by the sets of the form:

$W_F^{\epsilon'} := \{x \in B \mid |f_i(x)| < \epsilon', \ \forall i = 1, \ldots, p\}$, $\epsilon' > 0$ and $F = \{f_1, \ldots, f_p\} \subset \ell_1$.

LEMMA 6.1. *Let K be a convex,* $\sigma(\ell_\infty, \ell_1)$*–compact and circled subset of* ℓ_1 *and* $\epsilon > 0$, *then there exists* $\epsilon' > 0$ *and* $F = \{f_1, \ldots, f_p\} \subset \ell_1$ *such that* $W_F^{\epsilon'} \subset V_K^\epsilon$.

PROOF OF LEMMA 6.1: Let U be a neighborhood of 0 for $\tau(\ell_\infty, \ell_1)$. Recall that, in view of Eberlein and Smulian Theorem (see p. 256 in ALIPRANTIS–BORDER [1]), compactness and sequentially compactness coincide on ℓ_1. By Theorem 9, in DUNFORD–SCHWARTZ [10] (p. 292), there exists $M > 0$ such that $K \subset M\overline{B_1}$ and $\forall \epsilon > 0$, $\exists N \in \mathbf{N}$ such that $\forall g \in K$, $|\sum_{k=N+1}^{+\infty} g_k| < \epsilon/2$. Moreover, by Corollary 10 in DUNFORD–SCHWARTZ [10] (p. 293), one has $\forall \epsilon > 0$, $\exists N \in \mathbf{N}$ such that $\forall g \in K$, $\sum_{k=N+1}^{+\infty} |g_k| < \epsilon/2$. Let $g \in K$ and $x \in W_F^{\epsilon'}$. I define $F := \{(f_k)_{k\in\mathbf{N}} \mid f_k \in \{-1, 1\} \ \forall k \leqslant N; f_k = 0 \ \forall k > N\}$. Hence, $|g(x)| = |\sum_{k=0}^{+\infty} g_k x_k| \leqslant \sum_{k=0}^{N} |g_k||x_k| + \sum_{k=N+1}^{+\infty} |g_k||x_k| \leqslant \sum_{k=0}^{N} |g_k||x_k| + \frac{\alpha\epsilon}{2} \leqslant M \sum_{k=0}^{N} |x_k| + \frac{\alpha\epsilon}{2}$. Since $x \in W_F^{\epsilon'}$, one has $\sum_{k=0}^{N} |x_k| < \epsilon'$. Then, $|g(x)| \leqslant M\epsilon' + \frac{\alpha\epsilon}{2}$. Setting $\epsilon' = \frac{2-\alpha}{M}\epsilon$, one has the conclusion of Lemma 6.1.