ANNALES D'ÉCONOMIE ET DE STATISTIQUE. – N° 78 – 2005

Plausibility of Indeterminacy and Complex Dynamics

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ABSTRACT. – We study a pure exchange economy with infinite-lived agents, in which a share of consumption purchases must be paid cash. The economy may exhibit multiple equilibria, no matter what the fundamental specification, the only requirement being a share of consumption to be paid cash sufficiently low. Complex dynamics, such as chaos and cycles of any periodicity, can emerge under gross substitutability, a condition usually known for eliminating such phenomena.

Sur la plausibilité de l'indétermination et des dynamiques complexes

RÉSUMÉ. – Nous caractérisons une économie d'échange pur avec agents à durée de vie infinie, où une partie des achats doit obligatoirement se faire en encaisses préalables. Des équilibres multiples apparaissent dans cette économie pour n'importe quelle spécification des fondamentaux, pourvu que, étonnamment, l'imperfection du marché du crédit soit infime. Des dynamiques complexes, des cycles de toute période et du chaos, peuvent surgir, même sous l'hypothèse de substituabilité, une condition normalement connue pour éliminer ces phénomènes.

We are grateful to Jess BENHABIB, Gaetano BLOISE, Jean-Pierre DRUGEON, Jean-Michel GRANDMONT, Roger GUESNERIE, Michel GUILLARD, Alain VENDITTI for helpful comments. We are deeply indebted in particular with two anonymous referees for their decisive suggestions. We wish also to thank participants to the Summer School in Economic Theory organized by SET at Venice International University (June 30 – July 7, 2002). The usual disclaimers apply.
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1 Introduction

The main criticism addressed towards indeterminate models points out the weak support such models receive from empirical studies estimating the degree of increasing returns exploited or the unconventional calibrations for fundamental specifications adopted.

Early indeterminate models by Benhabib and Farmer [1994] and Farmer and Guo [1994] require amount of increasing returns too high to fall within standard errors of the empirical estimates of Basu and Ferland [1997] in which return to scale seems to be essentially constant. More recent theoretical works propose alternative models in which indeterminacy can be obtained at more plausible degrees of increasing returns (see, e.g., Wen's [1998] one-sector model with variable capacity utilization), although they remain strictly larger than unity. One way exploited to drive returns arbitrarily close to one, consists in shifting investigation from one-sector models to multi-sector models (e.g. Benhabib and Farmer, [1996]; Benhabib and Nishimura, [1998]; Venditti, [1998]), at the costs of being unable to replicate stylized facts of the *US* business cycle, at least when solely driven by beliefs shocks. Alternative successful ways include, among the others, Cazzavillan [2001] *OLG* models which relies upon an elasticity of labor supply arbitrarily large.

Another class of indeterminate models introduces frictions in capital market due to the presence of money as a medium of exchange. Yet, their success appears to rely upon unconventional fundamental specifications: To provide some examples, in Barinci and Chèron [2001] cash-in-advance economy, indeterminacy requires very strong income effects in intertemporal consumption; Farmer [1997] model with money in the utility function deals with non-standard preferences and Benhabib and Farmer [2000] introduce money as an argument in the production function. In Woodford [1986] and Grandmont *et al.* [1998] models with heterogenous agents and financial constraint, multiple equilibria require arbitrarily low elasticities of factors substitutability.

Subject to analogous skepticism is the emergence of complex dynamics and chaotic behavior: Since the seminal contribution of Grandmont [1985], it seems to be linked principally to *OLG* structures and to require strong complementariness in consumption and leisure, features not fully satisfactory on empirical ground.

In this paper we challenge these criticisms and show that indeterminacy, sunspot equilibria as well as complex dynamics such as chaos and cycles of any periodicity are phenomena the nature of the which may be very pervasive: They do rely neither upon unconventional, if not even implausible, fundamental specifications nor they require the limited participation characterizing *OLG* models. To show this, we consider a simple monetary pure exchange economy with representative agent *à la* Lucas and Stokey [1987], in which a given share of consumption purchases must be paid cash in the hand of the representative consumer. We find that indeterminacy and sunspot equilibria are compatible with whatever fundamental specification, the only requirement being an amount of consumption to be paid cash sufficiently low. Chaos and cycles of any periodicity emerge under gross substitutability, condition often known for ruling out such phenomena. Again, the crucial parameter is the amplitude of the liquidity constraint. Such a parameter has a specific economic interpretation: It represents the reciprocal of the income velocity of circulation of money, according to the Cambridge Cash Balance approach. Thus, focusing on a partial cash-in-advance constraint does not consist in a merely *ad hoc* device, but is fully justified on an appropriate theoretical ground.

Bosi and Magris [2003] and Bosi *et al.* [2002] find analogous results in terms of indeterminacy in an economy with capital accumulation. Nevertheless, the three-dimensional nature of the dynamic system obtained prevent them to go beyond a purely local analysis. Conversely, our relatively simple one-dimensional equation describing intertemporal equilibrium allows us to perform a global analysis too.

As it is shown in Grandmont *et al.* [1998] and Cazzavillan [1996], the occurrence of indeterminacy implies the existence of sunspot equilibria near the steady state as well as along bifurcations. Global analysis requires a reconsideration of sunspot equilibria: They can be constructed not only locally, but also globally, namely in any open and connected interval included between the autarkic steady state and the image of the maximum of the logistic map describing the backward dynamics of the economy. Indeed, the characterization of Grandmont *et al.* [1998] and Cazzavillan [1996] in terms of invariant support such that the system lies in it with probability one in each period, does still apply.

The emergence of complex dynamics allows to overcome the unpleasant properties characterizing analogous models with capital accumulation, as in Barinci and Chèron [2001], Bosi and Magris [2003] and Bosi *et al.* [2002], in which indeterminacy arises always through a two-period cycle, a feature hard to justify on empirical ground. In our model, we obtain cycles of any periodicity and chaotic behavior, which may better fit available data.

Most part of our analysis retains separable preferences. However a general formulation is also provided and analogous results in terms of indeterminacy are obtained. Thus, the separability assumption does not entail any loss of generality and is justified for its analytical tractability.

The sequel of the paper is organized as follows. In Section 2 we describe the economy and derive intertemporal equilibrium. Section 3 is devoted to the study of local indeterminacy and an economic interpretation of the phenomenon is provided. Moreover we generalize the local stability analysis by considering non-separable preferences, with particular attention to those compatible with balanced growth and deeply exploited in literature since the works of Kydland and Prescott [1982] and King, Plosser and Rebelo [1988]. In Section 4 we perform a global analysis, by focusing on isoelastic utility functions. Section 5 concludes the paper.

2 The environment and intertemporal equilibrium

We consider an infinite horizon economy with many identical infinite lived households whose size is normalized to unity. Preferences of the representative agent are given by the discounted sum of separable instantaneous utilities:[1]

(1) $$\sum_{t=0}^{\infty}\beta^t\left[u(c_t)-v(l_t)\right].$$

1. We consider later a general formulation with non-separable preferences.

The variables c and l stand, respectively, for consumption and labor and $\beta \in (0,1)$ for the discount factor. The per-period utility functions u and v possess the following features.

ASSUMPTION 1. *$u : R_+ \to R$ is smooth, strictly increasing and strictly concave with $\lim_{c\to 0^+} u'(c) = +\infty$. $v : R_+ \to R$ is smooth, strictly increasing and weakly convex. Moreover*

$$\lim_{y\to 0+} u'(y)/v'(y) > 1/\beta, \tag{2}$$

$$\lim_{y\to +\infty} u'(y)/v'(y) < 1 \tag{3}$$

Conditions in Assumption 1 are verified in a broad class of utility functions including the isoelastic specifications introduced in Section 5 when performing the global analysis. Such conditions are to ensure an interior solution for individual problem as well as the existence and uniqueness of a stationary solution. When maximizing (1), agents must take into account the dynamic budget constraint

$$M_{t+1} + B_{t+1} + p_t c_t = M_t + (1+i_t)B_t + p_t w_t l_t, \tag{4}$$

where M denotes money balances, B nominal bonds, p the price of consumption good, i the nominal interest rate on bonds holding, and w the real unit wage. The initial endowment of bond holding is zero: $B_0 = 0$. We assume that agents must pay cash at least a share $q \in (0,1]$ of their consumption purchases, *i.e.* they are subject to the additional financial constraint

$$qp_t c_t \leq M_t. \tag{5}$$

Notice that when constraint (5) binds, it represents the so-called equation of exchange in the Cambridge Cash Balance approach with q, the reciprocal of the income velocity of circulation, establishing the number of times each unit of money must circulate within a given period in order to finance economic transactions. The program of the representative agent consists in maximizing (1) subject to (4) and (5) and the usual transversality condition. Denoting λ and μ the Lagrangian multipliers associated to budget constraint and financial constraint, respectively, and assuming that constraint (5) binds[2] the FOC's write:

$$-\lambda_{t-1} + \lambda_t + \mu_t = 0, \tag{6}$$

$$-\lambda_{t-1} + \lambda_t (1+i_t) = 0, \tag{7}$$

$$\beta^t u'(c_t) - \lambda_t p_t - \mu_t q p_t = 0, \tag{8}$$

$$-\beta^t v'(l_t) + \lambda_t p_t w_t = 0. \tag{9}$$

2. Constraint (5) binds when the nominal interest rate is positive. In the sequel, we show that this is true at the steady state and thus nearby.

By manipulating conditions (6)–(9) and defining $\pi_t \equiv p_t/p_{t-1}$ the inflation factor between period t–1 and period t, we obtain the following useful expressions:

$$u'(c_t) = \beta u'(c_{t+1}) \frac{q(1+i_t)+(1-q)}{q\pi_{t+1}+(1-q)(1+i_{t+1})^{-1}\pi_{t+1}} \tag{10}$$

$$\beta u'(c_{t+1}) = (1-q)\frac{\beta v'(l_{t+1})}{w_{t+1}} + q\frac{v'(l_t)}{w_t}\pi_{t+1}, \tag{11}$$

Conditions (10)-(11) are arbitrage equations. The first one is the Euler equation and ensures optimal intertemporal consumption smoothing: The return of decreasing one unit of foregoing consumption in period t allows to increase consumption in period t+1 of an amount depending upon current and future nominal interest rate as well as inflation, and reflects the fact that a share of consumption requires previous investment in money balances. Condition (11) establishes that consumption in each period is financed in part out of current wage income and in part out of past one, after previous conversion in money balances.

We assume that one unit of labor can be used to produce one unit of output y according to the aggregate linear production function

$$y_t = L_t, \tag{12}$$

where L stands for aggregate labor. Finally, an exogenously given amount $\bar{M}$ of *fiat* money is available in the economy in each period[3].

Coupling equation (11) with equilibrium conditions, setting $U(y) \equiv u'(y)y$ and $V(y) \equiv v'(y)y$, and rearranging, we can fully describe a perfect foresight equilibrium as follows.

Definition 1 *(Intertemporal equilibrium) An intertemporal interior equilibrium with perfect foresight is a sequence* $\{y_t\}_{t=0}^{\infty}$, $y_t > 0$ *for all* $t \geq 0$, *satisfying*

$$q\beta^{-1}V(y_t) = U(y_{t+1}) - (1-q)V(y_{t+1}) \tag{13}$$

and the transversality condition.

A stationary solution of system (13) is a level of output $\bar{y}$ solving equation

$$u'(y)/v'(y) = 1-q+q/\beta. \tag{14}$$

Under Assumption 1, $u'(y)/v'(y)$ is decreasing in y: Thus boundary conditions (2)–(3) ensure existence and uniqueness of a interior steady state for any $q \in (0,1]$. It is easily verifiable that steady state welfare can be ranked according to the value of q, and that it decreases with the latter. Indeed, the right-hand side of (14) is increasing in q. The remainder of the proof follows directly from the concavity of u and the

3. Assuming government to peg the money growth at a constant rate would not change our results from a qualitative point of view.

weak convexity of v. From (11) and taking into account (14), one obtains that the stationary nominal interest rate is strictly positive:

$$\bar{\imath} = q^{-1}\left(\left[u'(\bar{y})/v'(\bar{y})\right]-1\right) = (1-\beta)/\beta > 0.$$

It follows that constraint (5) binds at the steady state, and thus in a neighborhood of it. Therefore system (13) is consistent, in the sense that it describes equilibria near the steady state. Finally, notice that at the steady state the price level is constant and equal to $\bar{M}/(q\bar{y})$.

3 Local analysis: Indeterminacy and bifurcations

In this section we aim at characterizing the stability of the steady state of system (13). For this purpose, it is useful to operate a change of variable by setting $x \equiv V(y)$ and then study the backward dynamics of the transformed difference equation

(15) $$x_t = \beta q^{-1}\left[W(x_{t+1}) - (1-q)x_{t+1}\right] \equiv F(x_{t+1})$$

where $W \equiv U \circ V^{-1}$. With $q = 1$, we obtain the equivalent of the standard *OLG* model with money, extensively studied by GRANDMONT [1985] and others, or of the infinite horizon economy à la LUCAS and STOKEY with cash good and credit good[4]. With $q = 0$, from (13) it easy to see that the dynamics of the economy degenerates, and we get the static condition $x = W(x)$ which means that the economy does not display any dynamics. Our interest focuses on the dynamics emerging when q is set between these two extremes. Output y in each period is a non-predetermined variable: Then the steady state of system (15) will be locally indeterminate if and only if it is stable in forward-looking dynamics, *i.e.* unstable in backward-looking. Formally, local indeterminacy requires $|F'(\bar{x})| > 1$, where $\bar{x} = V(\bar{y})$ is the unique interior stationary solution of (15). Straightforward computations yield the following expression for $F'(\bar{x})$:

(16) $$F'(\bar{x}) = \bar{\alpha} - \beta(1-\bar{\alpha})\left(q^{-1}-1\right)$$

where $\bar{\alpha} \equiv W'\,\bar{x}/W \in (-\infty, 1)$ is the elasticity of W evaluated at the fixed point. When consumption and leisure are gross substitutes, the domain of definition of $\bar{\alpha}$ is (0,1), otherwise, under gross complementarity, $\bar{\alpha} \in (-\infty, 0)$. Notice that $\bar{\alpha}$ represents the inverse elasticity of the offer curve $c = (U^{-1} \circ V)(l)$ introduced,

4. In our case consumption plays the role of cash good (although imperfectly, in view of the partial liquidity constraint), whilst leisure can be interpreted as credit good, since its purchases do not require money balances. Therefore, in spite of the presence of production, our model can be assimilated to a pure exchange economy with (partial) cash good and credit good.

among the others, by GRANDMONT *et al.* [1998] and CAZZAVILLAN *et al.* [1998] in their finance constrained economies.

By a direct inspection of (16), one easily verifies that $F'(\bar{x})$ is increasing in q for all $\bar{\alpha}$. In particular $lim_{q \to 0^+} F'(\bar{x}) = -\infty$, $F'(\bar{x}) = \bar{\alpha}$ when $q = 1$ and $F'(\bar{x})$ is always negative for $\bar{\alpha} < 0$. In addition $F'(\bar{x}) = -1$ when q is equal to

(17) $$q_F \equiv \left[1 + \beta^{-1}(1+\bar{\alpha})/(1-\bar{\alpha})\right]^{-1}.$$

Notice that $q_F \in (0,1]$ if and only if $\bar{\alpha} > -1$. At the same time, $F'(\bar{x}) = 0$ when q goes through

(18) $$q_0 \equiv \left[1 + \beta^{-1}\bar{\alpha}/(1-\bar{\alpha})\right]^{-1}.$$

One easily verifies that $q_0 \in (0,1]$ if and only if $\bar{\alpha} > 0$. All these pieces of information are sufficient to establish that when $\bar{\alpha} \in (0.1)$, then $F'(\bar{x})$ belongs to $(-\infty,-1)$ for $0 < q < q_F$, to $(-1,0)$ for $q_F < q < q_0$ and eventually to $(0,1)$ for $q_0 < q \leq 1$. Conversely, when $\bar{\alpha} \in (-1,0)$, $F'(\bar{x})$ belongs to $(-\infty,-1)$ when $0 < q < q_F$ and to $(-1,0)$ otherwise. Finally, when $\bar{\alpha} < -1$, $F'(\bar{x}) \in (-\infty,-1)$ for all q's. All these results are summarized in the following Proposition.

PROPOSITION 2 *(Local stability). Let* $\bar{\alpha} > 0$ *(Gross substitutability). Then* $F'(\bar{x})$ *belongs to:*

(i) $(-\infty,-1)$ *for* $0 < q < q_F$. *Thus the steady state is stable in forward looking (locally indeterminate);*

(ii) $(-1,0)$ *for* $q_F < q < q_0$. *Thus the steady state is unstable (locally determinate);*

(iii) $(0,1)$ *for* $q_0 < q \leq 1$. *Thus the steady state is unstable (locally determinate);*

Let $-1 < \bar{\alpha} < 0$ *(Weak complementarity). Then* $F'(\bar{x})$ *belongs to:*

(iv) $(-\infty,-1)$ *for* $0 < q < q_F$. *Thus the steady state is stable (locally indeterminate);*

(v) $(-1,0)$ *for* $q_F < q \leq 1$. *Thus the steady state is unstable (locally determinate);*

Let $\bar{\alpha} < -1$ *(Strong complementarity). Then* $F'(\bar{x})$ *belongs to:*

(vi) $(-\infty,-1)$ *for all q. Thus the steady state is stable (locally indeterminate);*

Moreover the convergence is oscillatory in the cases (i), (iv) and (vi), and, when q goes through q_F, *a flip bifurcation does occur.*

Proposition 2 shows that under full cash-in-advance constraint, $q = 1$, indeterminacy requires strong complementarity between consumption and leisure, namely $\bar{\alpha} < -1$, a result in line with AZARIADIS [1981] and GRANDMONT [1985] findings in their *OLG* economies. When q is progressively relaxed, the range of admissible values for $\bar{\alpha}$ improves: Indeterminacy becomes compatible with progressively higher degrees of substitutability. The occurrence of a flip bifurcation at $q = q_F$ implies the existence of a two-period cycle for any q belonging to a sufficiently small neighborhood of q_F. Nevertheless, in order to detect the stability of the cycle, one must specify fundamentals, namely the utility functions u and v, an exercise that we carry out in the sequel of the paper.

Following analogous lines as in CAZZAVILLAN [1996] and GRANDMONT *et al.* [1998], it is possible to construct sunspot equilibria around an indeterminate steady state as well as along flip bifurcations. The characterization of sunspot equilibria is made in terms of an invariant set such that output lies in it with probability one; Such a set is actually proved to exist around an indeterminate steady state as well as near a stable two-period cycle.

3.1 Non-separable preferences

Our results in terms of indeterminacy do still hold when consumption and leisure are not separable. Consider the instantaneous utility function $v(c,l)$ satisfying standard first order and second order assumptions as well as opportune boundary ones for the existence of a stationary solution. By maximizing $v(c,l)$ subject to (4) and (5) and by combining first order conditions with equilibrium ones, we still obtain the order difference equation (15) with $x \equiv V(y)$ and $W \equiv U \circ V^{-1}$, but now, crucially, functions U and V are defined by $U(y) \equiv v_1(y,y)y$ and $V(y) \equiv - v_2(y,y)y$ respectively[5]. Clearly, if we compute the backward-looking derivative $F'(\bar{x})$ of (15) evaluated at the steady state under study, we get exactly the expression (16). Analogous reasoning as in the separable case previously treated yields the same results included in Proposition 2.

If we retain the specification $v(c,l) \equiv (1 - 1/\sigma)^{-1}c^{1-1/\sigma}v(l)$, $v' > 0$, $v'' > 0$, deeply exploited in literature since the works of KYDLAND and PRESCOTT [1982] and KING, PLOSSER and REBELO [1988] in order to get an equilibrium path of balanced growth[6], straightforward computations yield the following expression for the critical q_F:

$$q_F = \frac{\beta}{1+\beta}\left(1+\frac{1-1/\sigma}{\varepsilon_v}\frac{\varepsilon_v+1-1/\sigma}{\varepsilon_{v'}+2-1/\sigma}\right)$$

with $\varepsilon_v \equiv \bar{y}v'(\bar{y})/v(\bar{y}) > 0$ and $\varepsilon_{v'} \equiv \bar{y}v''(\bar{y})/v'(\bar{y}) > 0$ the elasticities of v and v', respectively.

3.2 Interpreting conditions for indeterminacy

According to Proposition 2, the likelihood of indeterminacy improves as soon as the amplitude of the liquidity constraint q decreases. We now provide an interpretation of the mechanism leading to indeterminacy, in terms of arbitrages between consumption and labor supply, and of the adjustment in prices in response to agents revised expectations. To this end, let us suppose that the system is in period t at the steady state and let us analyze under which conditions it is possible to construct an alternative equilibrium in which agents anticipate, say, a fall in next period price

5. $v_i(y,y)$, $i = 1,2$, denotes the derivative of v with respect its i-th argument.

6. The other usual specification for preferences compatible with long-run growth has the form $v(c,l) \equiv \ln c + v(l)$ and represents a particular case of the separable formulation studied in the paper.

level. Supposing that consumption and leisure are gross substitutes[7], let us now investigate how the supply of labor in period t does react in response to the expected increase in consumption in period t+1. If q is close to one, almost the whole labor income in period t is invested in money balances to buy next period consumption, and the main arbitrage concerns period t leisure and period t+1 consumption. It follows, in view of the gross substitutability assumption, that leisure in period t is driven down – and thus labor supply is driven up – although less than proportionally than the corresponding increase in period t+1 consumption. Now, recall that equilibrium in good market requires labor to equalize consumption. This implies that also current consumption must increase, although slightly, with respect to its steady state value, a condition requiring a small fall of p_t, namely lower than that of p_{t+1}. This will induce an explosive dynamics violating at the end the transversality condition.

Consider now the case where q is low. Here, in correspondence to the expected increase in period t+1 consumption induced by the fall in p_{t+1}, agents will react by substituting period t labor with period t+1 one, since now a large part of period t+1 consumption is financed out of period t+1 labor income. Namely, the lower q, the higher the contraction in period t labor supply. Then, if q is low enough, in order to preserve good market equilibrium, period t price level must be sharply driven up to reduce current consumption. As a consequence, the system will move back towards its steady state, although following an oscillatory path.

4 Global analysis: Complex dynamics, cycles and chaos

In the previous section we have analyzed the stability of the steady state of system (16). According to Proposition 2, the steady state may be locally indeterminate, and thus sunspot equilibria can be constructed nearby, no matter which the fundamentals specification, the only requirement being an amplitude of the financial constraint sufficiently small.

However, given the relative simple formulation of equation (15) describing the dynamics of the economy, we can go easily beyond a purely local analysis and perform a global analysis. In such a context, we show that both cycles of any periodicity and chaos can emerge under gross substitutability, condition usually known for eliminating such phenomena. Again, the crucial requirement for the emergence of complex dynamics is represented by the amplitude of the liquidity constraint q.

To carry out the global analysis, for sake of analytical tractability, we focus on the following isoelastic utility functions:

$$u(c) \equiv \left(c^{1-1/\sigma} - 1\right) / (1 - 1/\sigma); \; v(l) \equiv l^{1+\varepsilon} / (1+\varepsilon)$$

7. In the opposite case, as we have shown in Proposition 2, local indeterminacy may emerge for whatever q.

with $\sigma, \varepsilon \in (0,+\infty)$ the elasticity of intertemporal substitution in consumption and the inverse of the elasticity of labor supply, respectively. Setting

$$\alpha \equiv (1 - 1/\sigma) / (1 + \varepsilon) \in (-\infty, 1)$$

equation (15) boils down to

$$x_t = \beta q^{-1}\left[x_{t+1}^{\alpha} - (1-q)x_{t+1}\right] \equiv F\left(x_{t+1}\right) \tag{19}$$

It is immediate to see that equation (19) gives raise to a logistic map if and only if α is strictly positive: Otherwise, it is monotonically decreasing in $x > 0$. Therefore, complex dynamics may emerge under gross substitutability, in contrast to *OLG* models in which strong degrees of complementarity are usually required. As is shown in GRANDMONT [1985], a sufficient condition for the existence of a non-degenerate 3-period cycle is $F^3(x^*) < x^* < F(x^*)$, with F^3 the third-iterate of F. Since in the SARKOWSKI pre-ordering 3 has the highest order among all positive integers, the existence of a k-period cycle, with k positive integer, implies the existence of cycles of any periodicity. Moreover, from LI and YORKE [1975] Theorem, the existence of a non-degenerate 3-period cycle implies chaos. To prove the existence of a 3-period cycle, let us start by observing that when α is positive, F is strictly concave and the unique maximum x^* is given by

$$x^* = \left[\alpha/(1-q)\right]^{1/(1-\alpha)}$$

The steady state of (19) is easily derived as

$$\bar{x} = \left[1 + q\left(\beta^{-1} - 1\right)\right]^{-1/(1-\alpha)}$$

As we have already seen, $F'(\bar{x}) < 0$ if and only if $q < q_0$. Thus, under the domain of latter inequality, $x^* < \bar{x}$ and as a consequence $x^* < F(x^*)$. We now turn to provide sufficient conditions for $F^3(x^*) < x^*$. Straightforward although tedious computations show that inequality $F^3(x^*) < x^*$ writes:

$$\left[\beta q^{-1}F\left(x^*\right)G\left(x^*\right)\right]^{-(1-\alpha)} < 1 - q + q\beta^{-1}x^*\left[\beta q^{-1}F\left(x^*\right)G\left(x^*\right)\right]^{-1} \tag{20}$$

with $G(x^*) \equiv F(x^*)^{-(1-\alpha)} - (1-q)$. The positivity of $F^2(x^*)$ requires that of $G(x^*)$ or equivalently

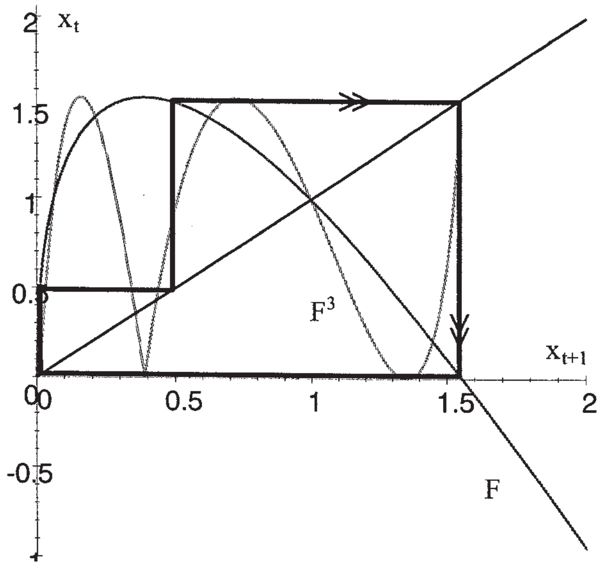
$$q > q_G \equiv \left[1 + \beta^{-1}(1-\alpha)^{-1}\alpha^{-\alpha/(1-\alpha)}\right]^{-1} \tag{21}$$

where q_G is solution of $G(x^*) = 0$. Since the right-hand side of (20) goes to $+\infty$ faster than its left-hand one for $q \to q_G^+$ when $\alpha \in (0,1)$, one has that in a right neighborhood of q_G inequality (20) is satisfied, jointly with the positivity of the second iterate $F^2(x^*)$. Finally notice that $q_G < q_0$. Therefore the sequence of x generated by the mapping F whenever it enters $[0,F(x^*)]$ will always remains in it. All this concludes the proof of the existence of chaos. All this is summarized in the following Proposition.

PROPOSITION 3 *(Global analysis). Let* $\alpha \in (0,1)$ *and* q_G *defined by (21). Then for q belonging to a right neighborhood of* q_G*, system (19) possesses a non-degenerate 3-period cycles. Hence cycles of any periodicity and chaos do emerge.*

To provide a quantitative example of a 3-period cycle (a fixed point for F^3), we refer to Fig. 1, where β has been set equal to 0.99, α to 1/2 and $q = 0.1983968$.

FIGURE 1



Since q represents the reciprocal of the velocity of circulation of money, it has a time dimension, like the discount factor β, and its value depends upon the length of the period considered. However, its plausible estimates do vary also in relation to the monetary aggregate one refers to. Therefore, there is a non-negligible range for its admissible values, even when the discount factor has been fixed.

Our global analysis shows that there may exist a closed interval $[0,F(x^*)]$ such that equation (19) takes values into $[0,F(x^*)]$, with $F(x^*) < \hat{x}$, where $\hat{x}$ is the positive zero of $F(x) = 0$. It follows that the characterization of sunspot equilibria can be reconsidered with respect to the case of a purely local analysis: As a matter of fact, any open connected interval included in $[0,F(x^*)]$ is a reliable support for constructing non-degenerate sunspot equilibria lying in $[0,F(x^*)]$ with probability one. Indeed, the characterization of sunspot equilibria in terms of invariant support made by GRANDMONT *et al.* [1998] and CAZZAVILLAN [1996] is of general applicability, although in their specific contexts it is exploited to construct sunspot equilibria only locally.

Global analysis can also be exploited to prove the existence of period-doubling flip bifurcations. When the steady state loses stability, a stable 2-period cycle emerges. Then, a 4-period cycle emerges when the 2-period cycle becomes unstable, and proceeding in this way cycles of higher periodicity arise as lower periodicity orbits lose stability[8].

8. The condition for the flip bifurcation occurring when the steady state of system (19) loses stability to be supercritical in forward looking is $3\varepsilon^3\sigma^2(4+\varepsilon)+4\varepsilon\sigma(2+3\varepsilon)(\sigma-1)+8(\sigma-1)(2\sigma-1) < 0$. (see BOSI and MAGRIS, [2002]).

5 Conclusion

We have presented a pure exchange infinite horizon model with a representative agent and partial liquidity constraint on consumption purchases. We have shown that the scope for indeterminacy improves as soon as the share of consumption good to be paid cash decreases, *i.e.* the income velocity of money circulation increases. Moreover, complex dynamics, such as cycles of whatever periodicity and chaotic behavior do emerge under gross substitutability, a condition usually known to rule out such phenomena. Results in terms of local indeterminacy do still hold for more general class of utility functions, included those compatible with long-run growth retained in the standard *RBC* literature.

In contrast to a widespread skepticism, our findings seem to suggest that indeterminacy and chaotic dynamics can be very pervasive phenomena, and they do not rely upon implausible increasing returns or controversial features such as strong consumption/leisure complementarity. ■

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