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Nonlinear Innovations and Impulse Responses with Application to VaR Sensitivity

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ABSTRACT. – This paper introduces impulse response analysis for nonlinear processes based on the concept of nonlinear innovation. Our approach borrows from the traditional linear impulse response analysis in that we consider shocks to innovations of a process. It also extends the methods of nonlinear impulse response analysis proposed earlier in the literature, in that it eliminates the problem of serial correlation of error terms, allows to examine permanent shocks, *i.e.* shocks occurring repeatedly in time, and provides straightforward interpretation of transitory or symmetric shocks. In our approach, the impulse responses are represented by the joint distribution of the perturbed and unperturbed paths. The analysis can be applied to processes such as the popular GARCH, or ACD models, and can be used to study shock sensitivity of dynamic financial strategies. As an illustration, we show how impulse responses can determine the Value at Risk and the minimum capital requirement under a dynamic portfolio management.

Innovations non-linéaires et fonctions réponses avec application à la sensibilité de la Valeur à Risque

RÉSUMÉ. – Nous introduisons un concept d'innovation adapté à l'analyse des dynamiques non linéaires. Nous expliquons comment le processus initial peut être exprimé en fonction des valeurs présentes et passées de l'innovation, utilisons les résidus associés pour construire des tests de spécification d'une dynamique non linéaire et pour définir des fonctions réponses à des chocs transitoires ou permanents. Il est expliqué pourquoi la distribution jointe des trajectoires perturbées et non perturbées est la représentation adéquate de la fonction réponse. Cette approche est illustrée sur des modèles dynamiques non linéaires du type ACD ou modèles à facteur. Elle est aussi utilisée pour étudier la Valeur à Risque et le capital requis dans le cas d'une stratégie dynamique de gestion de portefeuille.

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1 Introduction

The impulse response analysis is a widely used technique for exploring the dynamics of ARIMA models. It consists in tracing out and examining the response pattern of a model to external shocks. This pattern, called an impulse response function, is formally defined in the literature as the time profile of the shock effect on the behavior of the series (see, e.g. KOOP, PESARAN, POTTER [1996]). In recent literature, GALLANT, ROSSI, TAUCHEN [1993], and KOOP, PESARAN, POTTER [1996], (henceforth GRT and KPP) have pointed out essential differences between the impulse response analysis of linear and nonlinear processes. They can be briefly summarized as follows.

i) In ARIMA models, the impulse responses feature the property of symmetry (a transitory shock of -δ has exactly the opposite effect of a transitory shock of +δ), whereas for nonlinear processes the effects of shocks of opposite signs may be very different.

ii) The property of "shock linearity" (a transitory shock of $k\delta$ has k times the effect of a shock of δ) is satisfied in the linear framework whereas otherwise the effect of the magnitude of a shock is nonlinear.

iii) In ARIMA models the shock effect does not depend on the history of the process, while the path dependency is crucial in nonlinear framework.

iv) In nonlinear framework, it is necessary to consider the distributional properties of the impulse response function, besides tracing out the effect of a shock to the conditional expectation of the future variable of interest.

GRT and KPP have proposed extensions of the traditional impulse response analysis to nonlinear dynamic models. A common feature in GRT and KPP is the focus on impulse responses to transitory shocks, leaving out of scope the permanent shock analysis. Substantial differences between them are due to the adopted techniques. The approach proposed by GRT is conditioned on the observed history of the process, whereas KPP propose to integrate out the possible histories. Also, the transitory shocks are differently defined. In the spirit of the Keynesian multiplier analysis, GRT propose to shock directly the variable, instead of the innovation. The drawback of this approach is twofold. It can't be extended to the analysis of permanent shocks, which are shocks occurring repeatedly over an infinite time horizon. As well, the concept of symmetric shocks is unclear, since symmetric shocks to innovations do not necessarily correspond to symmetric shocks to the current variable. On the contrary, KPP following SIMS [1972] (see also BLANCHARD, QUAH [1989]), consider shocks to innovations. In particular, they study innovations obtained by conditionally centering and rescaling the variable Y_t. However, these standardized innovations:

$$v_t = (V_{t-1}Y_t)^{-\frac{1}{2}}(Y_t - E_{t-1}Y_t),$$

may not be sufficiently corrected for nonlinear temporal dependence. Indeed, they may feature temporal dependence in moments of order larger than two, which makes the interpretation of impulse responses unclear. As well, the conditional distribution of v_t is, in general, asymmetric. This also affects the interpretation of symmetric shocks.

The aim of the present paper is to improve upon the GRT and KPP methods in these respects. It is difficult to extend the linear impulse response analysis to a nonlinear framework due to an ambiguity in the definition of the innovation of a nonlinear process. Indeed, in the time series literature, innovations of a stochastic process (Y_t) are usually defined either as a) errors that represent differences between the expected and realized values of Y_t, *i.e.* $\epsilon_t^{(1)} = Y_t - E_{t-1}Y_t$, or b) conditionally standardized expectation errors, *i.e.* $\epsilon_t^{(2)} = (Y_t - E_{t-1}Y_t)/(V_{t-1}Y_t)^{\frac{1}{2}}$, where $E_{t\text{-}1}Y_t$ and $V_{t\text{-}1}Y_t$ are the conditional mean and variance of Y_t given the information available at time t-1. These definitions are not well suited for a study of nonlinear dynamics for the following reasons. The standardized innovations $(\epsilon_t^{(2)})$ are not necessarily independent, due to unobserved cross effects of their conditional moments of order strictly larger than two. Moreover, in a nonlinear framework, we are not only interested in the prediction of the variable Y_t itself, but also in the prediction of some nonlinear transformations of the variable. However, the definitions of innovations $\epsilon^{(1)}$ and $\epsilon^{(2)}$ are inadequate in this context. For example, $\epsilon_t^{(2)} = (Y_t - E_{t-1}Y_t)/(V_{t-1}Y_t)^{\frac{1}{2}}$ and $\epsilon_t^{(2)}(\exp) = (\exp Y_t - E_{t-1}\exp Y_t)/V_{t-1}(\exp Y_t)$ do not generally satisfy a deterministic one-to-one relationship. Thus, it is not possible to assimilate a shock to $\epsilon^{(2)}$ with a shock to $\epsilon_t^{(2)}(\exp)$. In the present paper, we introduce a notion of innovation for nonlinear processes which eliminates this ambiguity. Our response analysis borrows from the traditional approach in that it considers shocks to innovations. Its novelty consists in using the nonlinear innovations, instead of the traditional ones to trace impulse responses. As an illustration, we extend the domain of application of impulse response analysis to financial strategies, and consider shock effects to an outcome of a dynamic strategy of portfolio hedging.

In Section 2, we define a nonlinear Gaussian innovation of a strongly stationary process and derive a representation theorem for nonlinearly regular stationary process where the current value of the process is expressed as a nonlinear function of current and lagged nonlinear innovations. Section 3 concerns nonlinear impulse response analysis. We explain why the joint distribution of the perturbed and unperturbed paths is the appropriate representation for impulse responses. Next, we study the effects of permanent and transitory shocks to nonlinear Gaussian innovations and compare our approach to impulse response techniques introduced by Gallant, Rossi, Tauchen [1993], and Koop, Pesaran, Potter [1996]. Section 4 extends the impulse response techniques to a setup involving dynamic financial strategies. As an illustration, we discuss the structural impulse response analysis for determining the Value-at-Risk and the minimum capital requirement under a dynamic portfolio management. Section 5 concludes the paper.

2 Nonlinear Gaussian innovation and representation theorem

In this section, we define the nonlinear innovations and propose a representation theorem for nonlinear time series. The section also provides examples of nonlinear innovations for processes from the class of Nonlinear ARMA (NLARMA) processes.

Let us consider a univariate strongly stationary process ($Y_t, t \in Z$), and denote by $\mathcal{F}_t = \sigma(\underline{Y_t})$ the sigma-algebra generated by the current and past values of the process, where $\underline{Y_t} = (Y_t, Y_{t-1}, Y_{t-2} ...)$. Moreover, let us assume that the distribution of the current realization of the process conditioned on its past is continuous.

> Assumption A.1: *The conditional distribution of* Y_t *given* $\mathcal{F}_{t-1}$ *is continuous on* $[\mathcal{R}, \mathcal{B}(\mathcal{R})]$ *with a positive p.d.f. denoted by* f_{t-1}.

The conditional c.d.f. denoted F_{t-1} is continuous, strictly increasing and hence invertible.

2.1 Nonlinear Gaussian innovations

Intuitively, a nonlinear innovation (η_t) of the process (Y_t) is a strong white noise[1], such that the current value Y_t can be written as a function of its own past and of the current value η_t of the innovation process: $Y_t = g(\eta_t, \underline{Y_{t-1}})$, say, and Y_t and η_t satisfy a deterministic one-to-one relationship for a given $\underline{Y_{t-1}}$ [2]. Hence, if a nonlinear innovation exists, it is not uniquely defined, since any one-to-one transformation of a nonlinear innovation is also a nonlinear innovation. In this subsection, we prove the existence of a nonlinear innovation and show that the multiplicity of nonlinear innovations can be eliminated by introducing the identifiability condition of a Gaussian white noise.

Definition 1: The process $(\epsilon_t, t \in Z)$ is a nonlinear Gaussian innovation of the process ($Y_t, t \in Z$), if it satisfies the following conditions:

i) $(\epsilon_t, t \in Z)$ is a Gaussian white noise IIN(0,1);

ii) ϵ_t and Y_t satisfy a continuous invertible relationship conditional on $\mathcal{F}_{t-1}$: $\epsilon_t = g_{t-1}(Y_t)$ a.s., where g_{t-1} is continuous, invertible and may depend on the past $\mathcal{F}_{t-1}$.

> Proposition 1: *Under Assumption A.1, the strongly stationary process* ($Y_t, t \in Z$) *admits a nonlinear Gaussian innovation. It is unique up to a change of sign, date by date.*

Proof:

i) The process $(\epsilon_t, t \in Z)$ defined by:

$$\epsilon_t = \Phi^{-1}[F_{t-1}(Y_t)], t \in Z, \quad (2.1)$$

where Φ denotes the c.d.f. of the standard Normal distribution, satisfies the two conditions of Definition 1.

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1. that is a sequence of i.i.d. (zero mean) variables.
2. Generally, the standard innovations $\epsilon_t^{(1)}$ and $\epsilon_t^{(2)}$ presented in the introduction are not nonlinear innovations in this sense, since they do not satisfy the strong white noise property.

ii) Let us assume that ϵ_t^1 is another nonlinear Gaussian innovation. Then, there exists a continuous invertible relationship between ϵ_t and ϵ_t^1 conditional on $\mathcal{F}_{t-1}$: $\epsilon_t = h_{t-1}(\epsilon_t^1)$, (say), and $P[\epsilon_t^1 < \epsilon] = \Phi(\epsilon) = P[h_{t-1}(\epsilon_t^1) < \epsilon \mid \mathcal{F}_{t-1}]$.

The function $h_{t\text{-}1}$ is continuous, invertible on R and therefore monotone. If it is increasing, conditional on $\mathcal{F}_{t\text{-}1}$, we get:

$$\Phi(\epsilon) = \Phi[h_{t-1}^{-1}(\epsilon)], \ \forall \epsilon,$$

which implies $h_{t\text{-}1} = I_d$.

If it is decreasing, conditional on $\mathcal{F}_{t\text{-}1}$, we get:

$$\Phi(\epsilon) = 1 - \Phi[h_{t-1}^{-1}(\epsilon)], \ \forall \epsilon,$$

which implies $h_{t\text{-}1} = -I_d$.

Q.E.D.

Thus, it is always possible to choose a Gaussian innovation process such that the relationship between Y_t and ϵ_t is positive at any time t. This innovation is uniquely defined by (2.1). Moreover, formula (2.1) implies that the Gaussian nonlinear innovations of (Y_t) and of an invertible increasing transform of (Y_t) are identical. The choice of a Gaussian distribution is purely conventional, but has several advantages. First, it can be used to derive the Volterra representation of a nonlinear time series (see PRIESTLEY [1988] and Proposition 5 later in the text); second, the distribution of ϵ_t is symmetric, which is useful for examining symmetric shocks.

2.2 Representation theorem

This section introduces a representation theorem in which the current value of the process is written as a function of current and lagged values of innovations. By analogy to the WOLD representation of linear processes, we first discuss the nonlinear regularity condition.

Definition 2: The $(Y_t, t \in Z)$ process is nonlinearly regular if $\mathcal{F}_{-\infty} = \cap_t \mathcal{F}_t$ is empty.

In particular, this regularity condition implies that (GRANGER [1995])[3]:

$$\lim_{h \to \infty} E[a(Y_t, Y_{t+1}, \ldots, Y_{t+q}) \mid F_{t-h}] = Ea(Y_t, Y_{t+1}, \ldots, Y_{t+q}),$$

for any integer q and any integrable function a, for which the limiting expectation exists. This means that the initial value of the process does not help to predict over a long horizon.

PROPOSITION 2: *If the strongly stationary process* $(Y_t, t \in Z)$ *satisfies Assumption A.1 and is nonlinearly regular, then* $\sigma(\underline{\epsilon_t}) = \sigma(\underline{Y_t}) = \mathcal{F}_t$, where $\underline{\epsilon_t} = (\epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \ldots.)$, and $\underline{Y_t} = (Y_t, Y_{t-1}, Y_{t-2}, \ldots.)$.

3. It means that the process has short memory in-mean for any nonlinear transformation, using GRANGER's terminology (GRANGER [1995]).

PROOF: The second condition of Definition 1 implies:

$$\sigma(\epsilon_t, \mathcal{F}_{t-1}) = \sigma(Y_t, \mathcal{F}_{t-1}), \ \forall t, \tag{2.2}$$

and by recursion:

$$\sigma(\epsilon_t, \epsilon_{t-1}, \ldots, \epsilon_{t-p+1}, \mathcal{F}_{t-p}) = \sigma(Y_t, Y_{t-1}, \ldots, Y_{t-p+1}, \mathcal{F}_{t-p}), \ \forall t, \ p \geq 0. \tag{2.3}$$

i) We have: $\sigma(\epsilon_t, ..., \epsilon_{t-p}) \subset \sigma(\underline{Y_t}), \ \forall p,$
and then $\sigma(\underline{\epsilon_t}) = \vee_p \sigma(\epsilon_t, ..., \epsilon_{t-p}) \subset \sigma(\underline{Y_t}).$

ii) Conversely:

$$\sigma(\underline{Y_t}) = \sigma(\epsilon_t, ..., \epsilon_{t-p}) \vee \mathcal{F}_{t-p} \subset \sigma(\underline{\epsilon_t}) \vee \mathcal{F}_{t-p}, \ \forall p.$$

Therefore, $\sigma(\underline{Y_t}) \subset \cap_p [\sigma(\underline{\epsilon_t}) \vee \mathcal{F}_{t-p}] = \sigma(\underline{\epsilon_t}) \vee (\cap_p \mathcal{F}_{t-p}) = \sigma(\underline{\epsilon_t}),$ due to the regularity condition.

Q.E.D.

The representation theorem is a consequence of the existence of a simple Hilbert basis for Gaussian processes. More precisely, let us introduce the Hermite polynomials:

$$H_j(\epsilon) = \sum_{0 \leq m \leq [j/2]} \frac{j!}{(j-2m)!m!2^m} (-1)^m \epsilon^{j-2m} \quad j = 0,1,\ldots \tag{2.4}$$

A Hilbert basis of $L^2[\sigma(\underline{\epsilon_t})]$ is given by:

$$\frac{1}{\sqrt{j_1!}} H_{j_1}(\epsilon_{t-h_1}) \frac{1}{\sqrt{j_2!}} H_{j_2}(\epsilon_{t-h_2}) \ldots \frac{1}{\sqrt{j_n!}} H_{j_n}(\epsilon_{t-h_n}), \tag{2.5}$$

$n, j_1, \ldots, j_n, h_1, \ldots, h_n$ varying with $h_1 \neq h_2 \ldots \neq h_n$.

The representation theorem follows directly.

PROPOSITION 3: *If the strongly stationary process* (Y_t) *satisfies Assumption A.1, is nonlinearly regular and square integrable, we get:*

$$Y_t = \lim_{\substack{N \to \infty \\ J \to \infty \\ H \to \infty}} \sum_{n=1}^{N} \sum_{j_1, \ldots, j_n = 1, \ldots J} \sum_{\substack{h_1, \ldots, h_n = 0, \ldots, H \\ h_1 \neq \ldots \neq h_n}} a^{(N,J,H)}_{j_1, \ldots, j_n, h_1, \ldots, h_n} H_{j_1}(\epsilon_{t-h_1}) H_{j_2}(\epsilon_{t-h_2}) \ldots H_{j_n}(\epsilon_{t-h_n}),$$

where Y_t *is the mean square limit.*

This representation theorem is of Volterra type (see Volterra [1930], [1959], Nisio [1960], Priestley [1988]) and defines Y_t as a limit of polynomials in the current and lagged values of a Gaussian white noise. The traditional derivation of the Volterra expansion is based on Fourier transform arguments. Our approach concerns the time domain and is closer to the lines followed by Wiener [1958], using Hilbert arguments[4].

The condition of square integrability of Y_t is not very restrictive. Indeed, if Y_t is not square integrable, we may find an increasing transformation $h(Y_t)$, which will satisfy this requirement. Next, the representation theorem can be applied to the process $h(Y_t)$ and by inverting h, a representation for Y_t will be obtained (where Y_t becomes now the limit in probability).

2.3 Nonlinear ARMA Processes

The nonlinear innovations can be used to define a class of Nonlinear ARMA (NLARMA) models.

Definition 3: The strongly stationary process $(Y_t, t \in Z)$ has a nonlinear ARMA(p,q) or NLARMA(p,q) representation if, and only if, it satisfies a recursive relation (see Granger, Terasvirta [1993]):

$$Y_t = g(Y_{t-1}, \dots, Y_{t-p}, \epsilon_t, \dots, \epsilon_{t-q}), \tag{2.7}$$

where $(\epsilon_t, t \in Z)$ is a Gaussian nonlinear innovation, g is a function which is invertible with respect to ϵ_t, and is not constant with respect to Y_{t-p}, ϵ_{t-q}, respectively. The coefficients p and q are the (nonlinear) autoregressive and moving average orders, respectively.

i) **Nonlinear Autoregressive Process**

For $q = 0$, we get a nonlinear autoregression (NLAR) of order p (see Tong [1990], p. 96):

$$Y_t = g(Y_{t-1}, \dots, Y_{t-p}, \epsilon_t) \Leftrightarrow c(Y_t, Y_{t-1}, \dots, Y_{t-p}) = \epsilon_t, \text{ say.}$$

It is easy to check that a strongly stationary process admits a NLAR representation of order p if, and only if, it is Markov of order p.

Finally, note that the prediction error $\epsilon_t^{(1)}$ (resp. the conditionally standardized prediction error $\epsilon_t^{(2)}$) is a nonlinear innovation if the NLAR model can be written as:

$$Y_t = g(Y_{t-1}, \dots, Y_{t-p}) + h(\epsilon_t),$$

$$(\text{resp. } Y_t = g_0(Y_{t-1}, \dots, Y_{t-p}) + g_1(Y_{t-1}, \dots, Y_{t-p})h(\epsilon_t) \ ),$$

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4. The representation has an especially simple form, when the coefficients $a^{(N.J.H)}_{j_1,\dots,j_n,h_1,\dots,h_n}$ are independent of N,J,H. Indeed, we get:

$$Y_t = \sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=0}^{\infty} \sum_{h_1,\dots,h_n=0}^{\infty} a_{j_1,\dots,j_n,h_1,\dots,h_n} H_{j_1}(\epsilon_{t-h_1}) H_{j_2}(\epsilon_{t-h_2}) \dots H_{j_n}(\epsilon_{t-h_n}). \tag{2.6}$$

It is known that linear Gaussian ARMA models satisfy this condition whenever the moving average part does not admit a root with unitary modulus (Whittle [1963]).

where h is a one-to-one transformation. Thus, these standard definitions of innovations remain valid only in a restricted class of nonlinear dynamic models.

ii) **Nonlinear Moving Average Process**

When $p = 0$, we obtain a nonlinear moving average (NLMA) of order q (see TONG [1990], p. 115):

$$Y_t = g(\epsilon_t, \epsilon_{t-1} ..., \epsilon_{t-q}).$$

For a NLMA(q) process, the sigma algebras $\sigma(Y_t)$ and $\mathcal{F}_{t-q-1} = \sigma(\underline{Y_{t-q-1}})$ are independent. Therefore, we find zero autocorrelations between nonlinear transforms of current and lagged observations whenever the lag is large enough. This condition extends the common condition $Cov(Y_t, Y_{t-h}) = 0, \forall\, h \geq q+1$ valid in the linear framework. However, contrary to the linear case, it is not sufficient to characterize the nonlinear moving average process due to cross effects.

iii) **The ACD Model**

Let us now present an example of a mixed NLARMA(1,1) model. The traditional nonlinear dynamic models introduced for financial applications such as the Autoregressive Conditionally Heteroscedastic (ARCH) models (ENGLE [1982]) or the Autoregressive Conditional Duration (ACD) models (ENGLE, RUSSELL [1998]) usually contain nonlinear innovations in their specifications. As an illustration, let us consider the ACD(1,1) model, where the process of interest is a sequence of durations $\{Y_t, t \in \mathcal{Z}\}$. Let us introduce the conditional expectation of Y_t given the past: $\Psi_t = E(Y_t|\mathcal{F}_{t-1})$. It is assumed that the standardized durations Y_t/Ψ_t are independent with identical distributions whose c.d.f is F (say), and that Ψ_t satisfies the recursive equation:

$$\Psi_t = c + \alpha Y_{t-1} + \beta \Psi_{t-1}. \tag{2.8}$$

The nonlinear Gaussian innovation is:

$$\epsilon_t = \Phi^{-1}[F(Y_t / \Psi_t)]. \tag{2.9}$$

Therefore, we can write:

$$Y_t = \Psi_t g(\epsilon_t), \tag{2.10}$$

where $g = F^{-1}\Phi$, and by substituting into recursive equation (2.8) we get:

$$\begin{aligned} Y_t &= cg(\epsilon_t) + \alpha Y_{t-1} g(\epsilon_t) + \beta Y_{t-1} \frac{g(\epsilon_t)}{g(\epsilon_{t-1})} \\ &= \left[\alpha g(\epsilon_t) + \beta \frac{g(\epsilon_t)}{g(\epsilon_{t-1})} \right] Y_{t-1} + cg(\epsilon_t), \end{aligned} \tag{2.11}$$

which is an autoregressive representation with time dependent random autoregressive coefficients.

Recursive equation (2.8) can also be written as:

(2.12) $$\Psi_t = c + \alpha \Psi_{t-1} g(\epsilon_{t-1}) + \beta \Psi_{t-1},$$

which shows that the process (ϵ_t) is the nonlinear innovation of the expectation process (Ψ_{t+1}) as well. Therefore, (ϵ_t) is the common innovation process of the duration series, of its expectation and of the duration volatilities.

3 Impulse Response Analysis

The first part of this section reviews the methods developed by GRT and KPP. The remainder of this section presents our approach to the impulse response analysis, and provides analytical results on various shock dissipation patterns, illustrated by simulations.

3.1 The Definitions of Shocks in GRT and KPP impulse response analysis

Under the GRT approach, the nonlinear impulse response analysis involves a comparison of a conditional mean profile to a baseline profile for horizon j. More precisely, GRT consider a strictly stationary process (Y_t), and the baseline conditional mean profile given information up to time t $E[Y_{t+j} \mid Y_t, \underline{Y_{t-1}}]$, for $j = 1,2,\ldots$. Let δ^+, δ^- represent small positive and negative perturbation of fixed size in the vector of conditioning arguments. The impulse responses are formed by sequences $E[Y_{t+j} \mid Y_t + \delta^+, \underline{Y_{t-1}}] - E[Y_{t+j} \mid Y_t, \underline{Y_{t-1}}]$ and $E[Y_{t+j} \mid Y_t + \delta^-, \underline{Y_{t-1}}] - E[Y_{t+j} \mid Y_t, \underline{Y_{t-1}}]$. They represent the effects of a positive (resp. negative shock) on a given conditional mean at different lags.

This approach can be extended by considering the effects of shocks on the expectations of nonlinear transformations of the process as $E[g(Y_{t+j}) \mid Y_t, Y_{t-1}]$. By considering such transformations the nonlinear dynamics is taken into account. However, the shock δ to Y_t is hard to interpret, because it is assumed that it can be considered independently of the past. In contrast, under the KPP approach, a nonlinear regression model with conditional heteroscedasticity is assumed:

$$Y_t = m(\underline{Y_{t-1}}) + \sigma(\underline{Y_{t-1}})\epsilon_t,$$

and the effects of shock δ to the conditionally standardized residual $\epsilon_t = [Y_t - m(\underline{Y_{t-1}})]/\sigma(\underline{Y_{t-1}})$ are examined. Thus the shock hits the innovation rather than the variable Y itself. However, the conditionally standardized residual can be viewed as a nonlinear innovation only if the sequence (ϵ_t) is an iid sequence. Otherwise, this type of shock analysis disregards the dynamics of higher order moments, especially the dynamics of extremes, which is of much importance in risk analysis.

3.2 The impulse response function

In our approach, the impulse response function measures the effect of a sequence of shocks to a nonlinear innovation process on future values of the process of interest. Since the nonlinear innovation is not uniquely defined, we expect to find different impulse response functions depending on the selected nonlinear innovation. We first consider the impulse response function for the Gaussian nonlinear innovation. Next, we extend this basic concept to a general impulse response function.

i) **Gaussian Nonlinear Innovation**

The impulse response analysis can be based on the Volterra decomposition (see, Proposition 3), where:

(3.1) $$Y_t = a(\epsilon_t, \epsilon_{t-1}, ..., \epsilon_1, \underline{\epsilon_0}),$$

(ϵ_t) is a Gaussian white noise, with unit variance, and $\underline{\epsilon_0} = (\epsilon_0, \epsilon_{-1}, ...)$. Since the distribution of (ϵ_t) is symmetric, the shocks of δ and $-\delta$ have the same infinitesimal occurrence. Thus, we can compare the effects of shocks with identical occurrences, called symmetric shocks in the literature. As well, since the distribution of (ϵ_t) is independent of time, the shocks of the same magnitude δ at different dates also have the same infinitesimal occurrence, which allows us to consider the so-called "permanent" shocks.

As suggested by GRT [1993], the analysis needs to be conditioned on the history prior to the shock arrival. Therefore, if the shocks hit the process at date 1, the previous values of the process and innovations are known, that is $\underline{\epsilon_0}$ is fixed. Then, at date 0, we have to evaluate the effect of a sequence of deterministic shocks $\delta_1, \delta_2, ..., \delta_t ...$ to $\epsilon_1, ..., \epsilon_t, ...$ (occurring at future dates) on the future profile of the process (Y_t). These effects have to be measured with respect to a benchmark which is the path of (Y_t) in the absence of shocks. Since future innovations are unknown, this benchmark is random. We denote by: $\epsilon_1^s, \epsilon_2^s, ..., \epsilon_t^s, ...$ a future path for the innovations, where $\epsilon_1^s, \epsilon_2^s, ..., \epsilon_t^s, ...$ are IIN (0,1) conditional on $\underline{\epsilon_0} = (\epsilon_{-1}, \epsilon_{-2}, ...)$. The random benchmark is:

(3.2) $$Y_t^s\left(\underline{\epsilon_0}\right) = a\left(\epsilon_t^s, \epsilon_{t-1}^s, ..., \epsilon_1^s, ... \underline{\epsilon_0}\right),$$

whereas the profile after the shock arrival is:

(3.3) $$Y_t^s\left(\underline{\delta}, \underline{\epsilon_0}\right) = a\left(\epsilon_t^s + \delta_t, \epsilon_{t-1}^s + \delta_{t-1}, ..., \epsilon_1^s + \delta_1, \underline{\epsilon_0}\right),$$

where $\underline{\delta} = (\delta_1, ..., \delta_t, ...)$.

Definition 4: The entire effect of the sequence of shocks is summarized by the joint path distribution of:

$$[Y_t^s(\underline{\epsilon_0}), Y_t^s(\underline{\delta}, \underline{\epsilon_0}), t \geq 1].$$

and is called the **nonlinear impulse response**.

In practice, various sequences of shocks can be examined. The standard impulse response analysis concerns:

- either transitory shocks at date 1: $\delta_1 = \delta$, $\delta_t = 0$, $t \geq 2$,
- or transitory shocks at date t_0: $\delta_{t_0} = \delta$, $\delta_t = 0$, $t \neq t_0$,
- or permanent shocks starting at date 1: $\delta_t = \delta$, $t \geq 2$.

They differ in terms of the sign and magnitude of δ.

Note that it is common in the literature (GRT[1998], KPP[1996]) to summarize the response by considering only the summary statistics of the joint distributions, such as either the differences of expectations of the series:

$$E[Y_t^s(\underline{\delta}, \underline{\epsilon_0}) \mid \underline{\epsilon_0}] - E[Y_t^s(\underline{\epsilon_0}) \mid \underline{\epsilon_0}],$$

or the differences of expectations of the transformed series:

$$E[g(Y_t^s(\underline{\delta}, \underline{\epsilon_0})) \mid \underline{\epsilon_0}] - E[g(Y_t^s(\underline{\epsilon_0})) \mid \underline{\epsilon_0}],$$

where g is a given nonlinear function,
or the differences of variances:

$$V[Y_t^s(\underline{\delta}, \underline{\epsilon_0}) \mid \underline{\epsilon_0}] - V[Y_t^s(\underline{\epsilon_0}) \mid \underline{\epsilon_0}].$$

These approaches disregard the dependence between the benchmark and perturbed paths, accounted for by the joint distribution. Also note that in the nonlinear framework the perturbed path cannot be factorized, in general, as $Y_t^s(\underline{\delta}, \underline{\epsilon_0}) = f_t(\delta) Y_t^s(\underline{\epsilon_0})$, even for small δ. Therefore, the traditional multiplier effect $f_t(\delta)$ is meaningless in a nonlinear framework.

ii) **General Nonlinear Innovation**

Let us now consider another nonlinear innovation process: $\eta_t = h(\epsilon_t)$, say, where h is an invertible increasing function. We can write:

$$\begin{aligned} Y_t &= b\left(\eta_t, \eta_{t-1,\ldots,} \eta_{1,} \underline{\eta_0}\right) \\ &= a\left[h^{-1}(\eta_t), h^{-1}(\eta_{t-1}), \ldots, h^{-1}(\eta_1), \underline{h^{-1}(\eta_0)}\right]. \end{aligned}$$

For clarity of exposition, let us consider a transitory shock δ_1^* at date 1 to the innovation η_1. The profile after the shock arrival is:

$$Y_t(\delta) = a\left[h^{-1}(\eta_t), h^{-1}(\eta_{t-1}), \ldots, h^{-1}\left(\eta_1 + \delta_1^*\right), \underline{h^{-1}(\eta_0)}\right].$$
$$= a\left[h^{-1}(\eta_t), h^{-1}(\eta_{t-1}), \ldots, h^{-1}(\eta_1) + h^{-1}\left(\eta_1 + \delta_1^*\right) - h^{-1}(\eta_1), \underline{h^{-1}(\eta_0)}\right].$$

The deterministic shock to the nonlinear innovation η_1 is identical to a stochastic shock of magnitude: $\delta_1 = h^{-1}(\eta_1 + \delta_1^*) - h^{-1}(\eta_1) = h^{-1}[h(\epsilon_1) + \delta_1^*] - \epsilon_1$, to the Gaussian nonlinear innovation ϵ_1. Thus, a shock to the innovation can be deterministic or stochastic, depending on the selected nonlinear innovation.

Remark: A similar argument holds for a deterministic drift of the observation Y_1 itself (see, GRT [1993]). A deterministic drift of Y_1 corresponds generally to a stochastic drift of any nonlinear innovation η_1. Deterministic drifts of Y_1 and η_1 are compatible if and only if we can write $Y_1 = b_0(\underline{\eta_0}) + b_1(\underline{\eta_0})\eta_1$, that is, if the current innovation has a linear effect conditional on the past.

In fact, it is possible to reconcile the different definitions of shocks only by interpreting the impulse as a shock to the distribution of the innovation process and not as a shock to the innovation value. In the case of a Gaussian nonlinear innovation process considered in this paper, an impulse shifts horizontally the standard Normal distribution N(0,1) by δ, yielding the distribution N(δ,1). Therefore, it can be viewed as a drift impulse. However, other impulses would also be admissible, such as the one that rescales the standard Normal by δ, transforming the N(0,1) into N(0,δ). It could be interpreted as a shock to the innovation volatility, that is, a volatility impulse.

A further discussion of the extended definition of shocks is beyond the scope of this paper. Consequently, our approach is focused on the drift impulses to Gaussian innovations.

3.3 **Nonlinear AR(1) process**

i) **The dynamics.**

As an illustration, let us consider a dynamic NLAR(1) model defined by:

(3.4) $$Y_t = g\left(Y_{t-1}; \epsilon_t\right) = g^{*(1)}\left(Y_{t-1}; \epsilon_t\right), \text{ say,}$$

where (ϵ_t) is a standard Gaussian white noise and g a function invertible with respect to ϵ. By recursion, we find the expression of Y_t as a function of Y_0 and the innovation sequence ${}_t\epsilon_1 = (\epsilon_1, \ldots, \epsilon_t)$:

(3.5) $$Y_t = g^{*(t)}\left(Y_{0,t}; \epsilon_1\right),$$

where $g^{*(t)}$ is recursively defined by:

$$g^{*(t)}[Y_0,{}_t\epsilon_1] = g\{g^{*(t-1)}[Y_0,{}_{t-1}\epsilon_1],\epsilon_t\} \tag{3.6}$$

$$= g^{*(t-1)}\{g[Y_0,\epsilon_1],{}_t\epsilon_2\}. \tag{3.7}$$

The equality (3.6) implies:

$$\frac{\partial g^{*(t)}}{\partial y}[Y_0,{}_t\epsilon_1] = \frac{\partial g}{\partial y}[Y_{t-1},\epsilon_t]\frac{\partial g^{*(t-1)}}{\partial y}[Y_0,{}_{t-1}\epsilon_1],$$

and by the chain rule:

$$\frac{\partial g^{*(t)}}{\partial y}[Y_0,{}_t\epsilon_1] = \prod_{r=1}^{t}\frac{\partial g}{\partial y}[Y_{r-1},\epsilon_r]. \tag{3.8}$$

ii) **Local effect of a transitory shock at date 1.**
Let us consider a small transitory shock $\delta_1 = \delta$. We get:

$$\begin{aligned} Y_t(\delta) &= g^{*(t)}[Y_0,{}_t\epsilon_1 + \delta_1^t] \\ &= g^{*(t-1)}[g(Y_0,\epsilon_1+\delta),{}_t\epsilon_2], \quad \text{from (5.7)}, \\ &\approx g^{*(t-1)}[Y_1,{}_t\epsilon_2] + \frac{\partial g^{*(t-1)}}{\partial y}[Y_1,{}_t\epsilon_2]\frac{\partial g}{\partial y}[Y_{t-1},\epsilon_t]\delta \\ &= Y_t + \prod_{\tau=1}^{t}\frac{\partial g}{\partial y}[Y_{\tau-1},\epsilon_\tau]\delta, \quad \text{from (3.8)}. \end{aligned}$$

For a nonlinear transformation of the process, H say, we get:

$$H[Y_t(\delta)] \approx H(Y_t) + \frac{dH}{dy}(Y_t)\prod_{r=1}^{t}\frac{\partial g}{\partial y}[Y_{r-1},\epsilon_r]\delta. \tag{3.9}$$

Therefore, the infinitesimal effect of the transitory shock on the expectation of $H(Y_t)$ conditional on Y_0 is:

$$\tfrac{1}{\delta}E_0\{H[Y_t(\delta)] - H(Y_t)\} = E_0\left[\frac{dH}{dy}(Y_t)\prod_{r=1}^{t}\frac{\partial g}{\partial y}[Y_{r-1},\epsilon_r]\right]. \tag{3.10}$$

It is interesting to consider the long run impact of a transitory shock, that is, the behaviour of either a) $\frac{1}{\delta}[Y_t(\delta) - Y_t]$, or b) $\frac{1}{\delta}E_0[Y_t(\delta) - Y_t]$, when t tends to infinity. Indeed, NELSON [1990] (see also BOUGEROL, PICARD [1992]) has shown in the framework of GARCH models that long run responses to shocks can be significantly different depending if we consider them path by path (expression a)) or if we average over the paths (expression b)). More precisely, if the process is nonlinearly regular, then it has short memory (see Definition 2), and the shock effect dissipates in the long run. Therefore, asymptotically, the effect of a temporary shock to Y_t dies out to zero. For large t, this effect is:

$$\prod_{r=1}^{t} \frac{\partial g}{\partial y}\left[Y_{r-1,}\epsilon_r\right]$$
$$= \exp\left\{\sum_{r=1}^{t}\left[\frac{\log \partial g}{\partial y}(Y_{r-1},\epsilon_r)\right]\right\}$$
$$\sim \exp\left[tE\log\frac{\partial g}{\partial y}(Y_{r-1},\epsilon_r)\right],$$

and tends to zero if $E\log\frac{\partial g}{\partial y}(Y_{r-1},\epsilon_r)<0,$ which is a necessary condition for a strongly stationary regular process, since $E\log\frac{\partial g}{\partial y}(Y_{r-1},\epsilon_r)$ is the LIAPUNOV exponent of the dynamic system (OSELEDEC [1968]).

However, by taking the expectation and using Jensen inequality, we get:

$$E\left[\prod_{r=1}^{t}\frac{\partial g}{\partial y}\left(Y_{r-1,}\epsilon_r\right)\right]$$
$$= E\left[\exp\sum_{r=1}^{t}\log\frac{\partial g}{\partial y}(Y_{r-1},\epsilon_r)\right]$$
$$\geq \exp\left[E\sum_{r=1}^{t}\log\frac{\partial g}{\partial y}(Y_{r-1},\epsilon_r)\right]$$
$$\approx \exp\left[tE\log\frac{\partial g}{\partial y}(Y_{r-1},\epsilon_r)\right],$$

and the stationarity condition $E\left[\log\frac{\partial g}{\partial y}(Y_{r-1},\epsilon_r)\right]<0$ does not necessarily imply that the impulse response vanishes in average for large t.

iii) **Linear AR(1) model with a random autoregressive coefficient.**

Let us consider the dynamic bilinear model (Tong [1993], p. 8):

$$Y_t = (a+b\epsilon_t)Y_{t-1}+\epsilon_t, \quad t \text{ varying},$$

where (ϵ_t) is a standard Gaussian white noise. We can explicitly compute the effect of a transitory shock that occurs at date 1. The perturbed path is such that:

$$Y_t^D = (a+b\epsilon_t)Y_{t-1}^D+\epsilon_t, \quad \forall t \geq 2.$$

We see that:

$$\Delta Y_t = Y_t^D - Y_t = (a + b\epsilon_t)\Delta Y_{t-1} = \prod_{\tau=2}^{t}(a + b\epsilon_\tau)(1 + bY_0)(\delta\epsilon_1).$$

This model satisfies the property of shock linearity because the effect of the shock is a linear function of $\delta\epsilon_1$. Moreover, we know from Bougerol, Picard [1992] that, for large t, the coefficient:

$$\begin{aligned} &= \prod_{\tau=2}^{t}(a + b\epsilon_\tau)(1 + bY_0) \\ &= \exp\left\{(t-1)\frac{1}{t-1}\sum_{r=2}^{t}\log(a + b\epsilon_\tau)\right\}(1 + bY_0) \\ &\sim \exp\left[(t-1)E\log(a + b\epsilon_\tau)\right](1 + bY_0), \end{aligned}$$

tends to zero if and only if $E\log(a + b\epsilon_\tau) < 0$. However, the effect of the shock on the expectation of Y_t is:

$$E[\Delta Y_t \mid Y_0] = a^{t-1}(1 + bY_0)\delta\epsilon_1.$$

This average effect tends to zero if |a| < 1, which is a more stringent condition than the negativity of $E\log(a + b\epsilon_\tau)$, due to Jensen inequality.

Example 3.1: The ACD(1,1) introduced in subsection (2.3) belongs to the class of linear AR(1) models with random autoregressive coefficients. Indeed, the expected durations satisfy:

$$\Psi_t = c + [\alpha g(\epsilon_{t-1}) + \beta]\Psi_{t-1},$$

and the impact of a transitory shock is such that:

$$\Delta\Psi_t = [\alpha g(\epsilon_{t-1}) + \beta]\Delta\Psi_{t-1}, \text{ for } t \geq 3.$$

This effect vanishes asymptotically in average if:

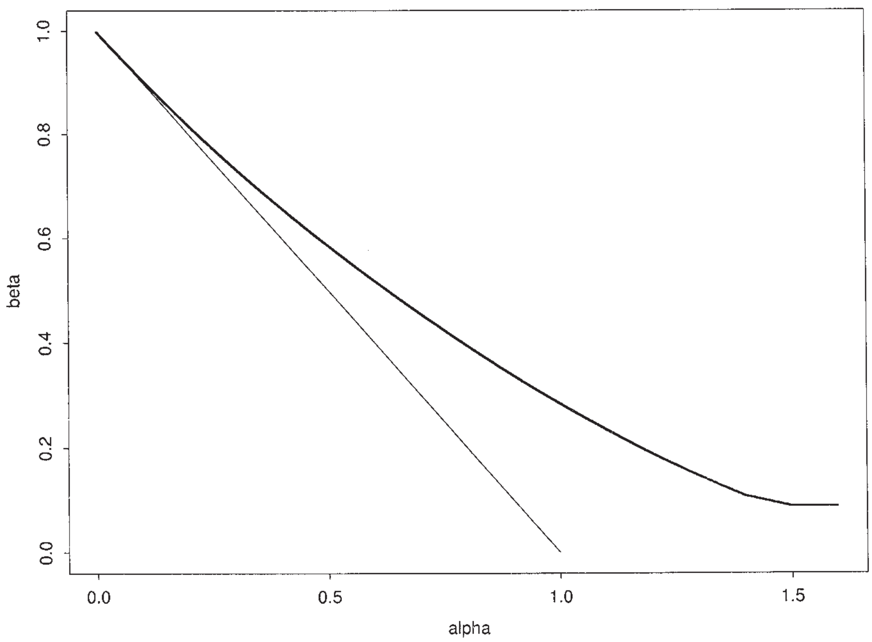
$$| E(\alpha g(\epsilon) + \beta) | = | \alpha + \beta | < 1,$$

and vanishes path by path if the Liapunov exponent is negative $E\log(\alpha g(\epsilon) + \beta) < 0$. These dissipation patterns depend on *α, β* and on the distribution of standardized durations. For instance, if $g(\epsilon)$ has an exponential distribution, the Liapunov exponent is:

$$\int_0^\infty \log(\alpha x + \beta)\exp(-x)dx = \log\beta + \exp(\beta/\alpha)E_1(\beta/\alpha),$$

where $E_1(x) = \int_0^\infty \frac{\exp -t}{t}\, dt$ is the exponential integral (see ABRAMOWITZ, STEGUN [1964], formula 5.1.1, page 228). We plot in Figure 3.1 the set of values (α,β) for which the Liapunov exponent is zero.

FIGURE 3.1
Liapunov Exponent Frontier



Below this frontier, all pairs of (α,β) coordinates are associated with negative Liapunov exponents, while the coordinates (α,β) above it are associated with its positive values. The frontier is decreasing from $(\alpha = 0.0,\ \beta = 1.0)$ down to $(\alpha \approx 1.4, \beta \approx 0.1)$. For β's greater than 1.4, the frontier approches asymptotically the α-axis.

3.4 **Simulation results**

In this section, we illustrate the computation and analysis of impulse response functions using as examples the ACD(1,1) model and a factor model with a specific form of nonlinear temporal dependence.

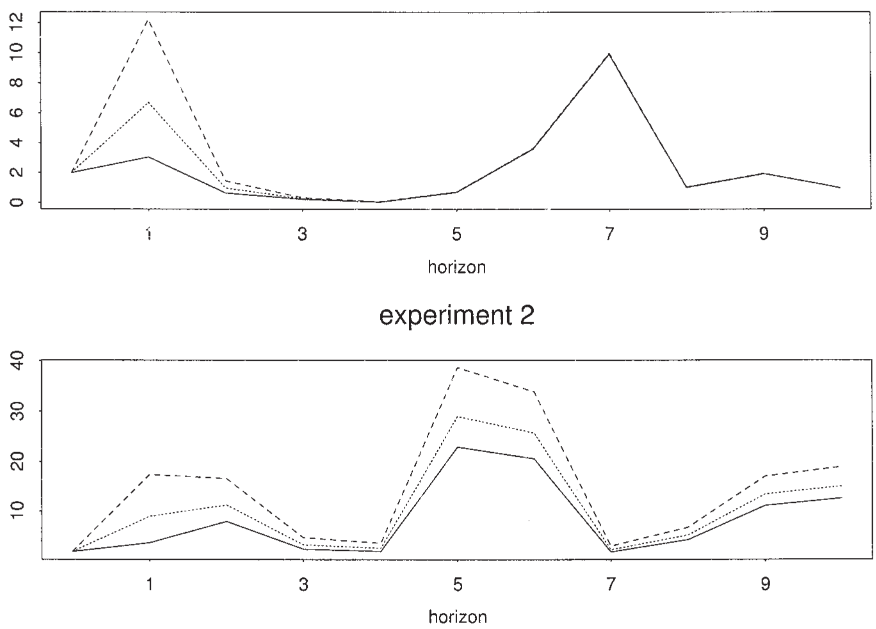
i) **The autoregressive conditional duration model.**

Let us consider the ACD(1,1) model of subsection (2.3) with an exponential distribution of the standardized durations. The initial values have been fixed to $\epsilon_0 = 0.0$, $y_0 = 2.0$. Two experiments are performed, for the following sets of parameter values:

- Experiment 1: $c = 1$, $\alpha = 0.3$, $\beta = 0.2$,
- Experiment 2: $c = 1$, $\alpha = 0.4$, $\beta = 0.64$.

In the first experiment, the shock effect vanishes asymptotically in the mean and path by path, whereas in the second experiment a different outcome is observed. We consider transitory shocks δ that occur at date 1, and take values δ = -1, -0.9,… 0.9,1, with the benchmark set at δ = 0.0. The maximum horizon is H = 10. Figure 3.2 displays the joint simulated paths for the benchmark and two perturbed series with δ = +1 and δ = -1, for both experiments. The effects of shocks quickly dissipate in Experiment 1 whereas they are more persistent in Experiment 2 although they also vanish asymptotically.

FIGURE 3.2
Simulated Paths



We plot in Figure 3.3 the (marginal) distribution of $Y_t(\delta)$ for horizon t = 3, and δ = -1,0,+1 conditional on the information available at date 0.

The shocks have an effect on both the means and tails of the distribution. These effects can be evaluated by considering:

- the mean deviation $EY_t(\delta)$ - EY_t,
- the variance of the deviation from benchmark: $V(Y_t(\delta) - Y_t)$,

for different horizons t = 1,…,10 and different values of transitory shocks. They are shown in Figures 3.4 and 3.5 for Experiment 1, and in Figures 3.6 and 3.7 for Experiment 2.

We observe explosive patterns of the averaged effects of shocks in the second experiment, although earlier no explosive paths were found. The patterns of responses associated with different shocks are similar due to the simple formula of the deviation from benchmark for the ACD(1,1) model. Indeed, it is easy to see that:

$$Y_t(\delta) - Y_t = \alpha \prod_{\tau=3}^{t} \left[\alpha g(\epsilon_\tau) + \beta \frac{g(\epsilon_\tau)}{g(\epsilon_{\tau-1})} \right] g(\epsilon_2) \left[c + \left(\alpha + \frac{\beta}{g(\epsilon_0)} \right) y_0 \right] [g(\epsilon_1 + \delta) - g(\epsilon_1)],$$

FIGURE 3.3
Marginal Distribution at Horizon 3

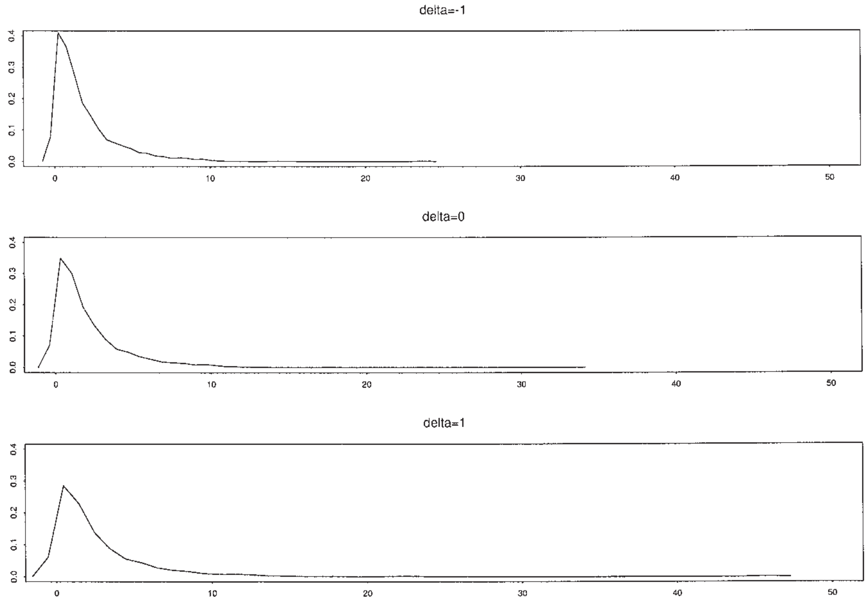


FIGURE 3.4
Mean Deviation from the Benchmark

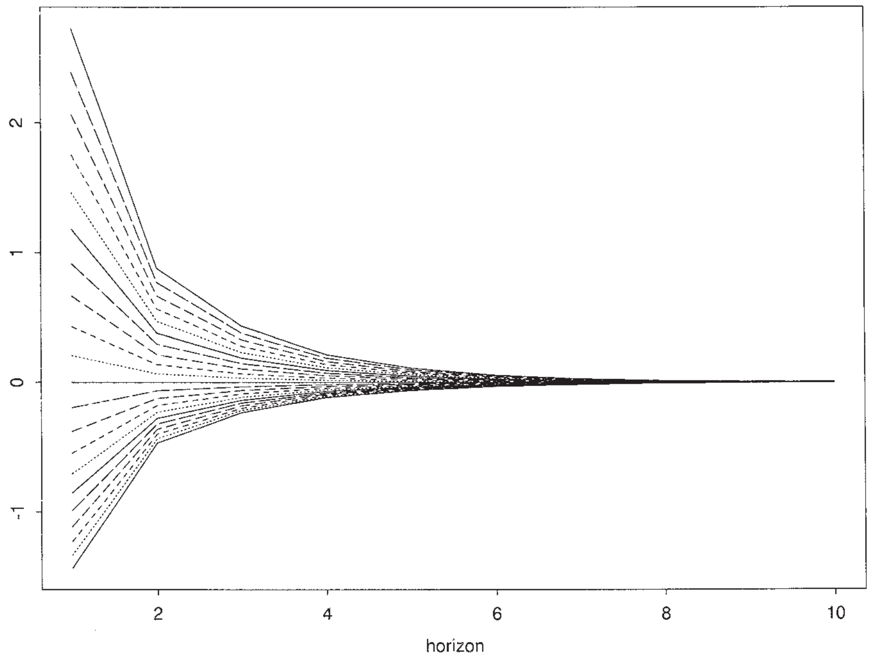


FIGURE 3.5
Variance of the Deviation from Benchmark

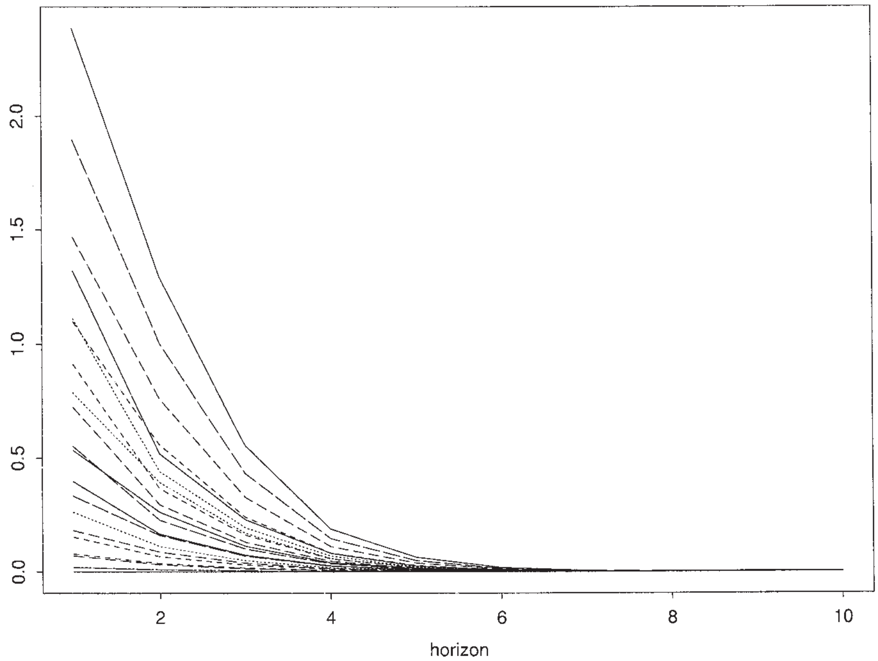


FIGURE 3.6
Mean Deviation from the Benchmark

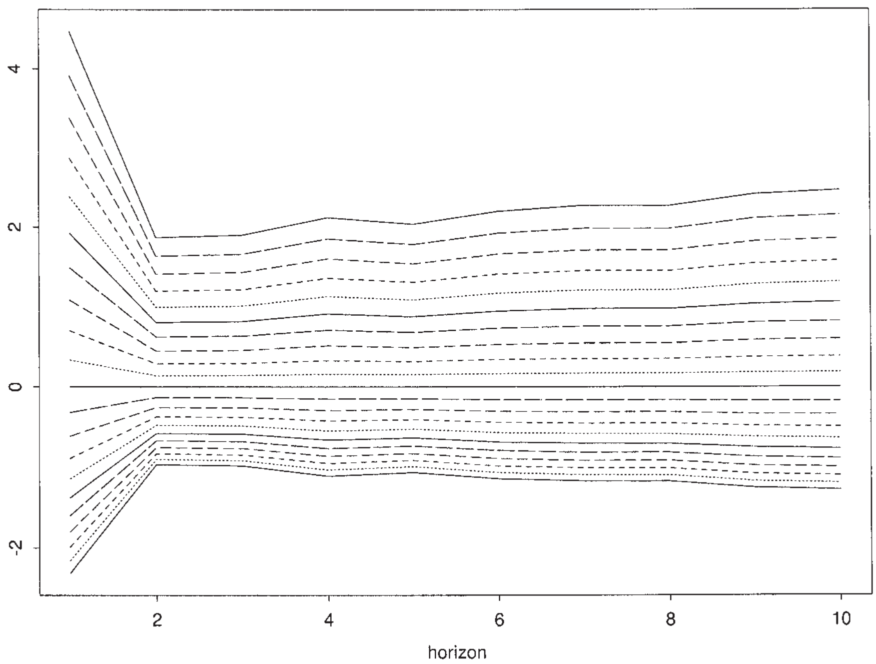
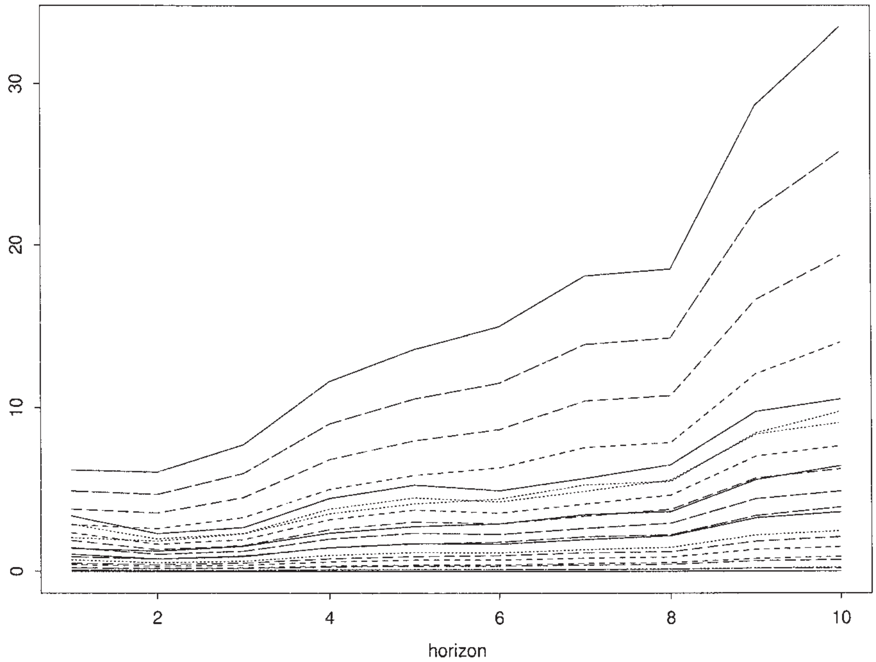


FIGURE 3.7
Variance of the Deviation from Benchmark



which implies:

$$E[Y_t(\delta) - Y_t] = A_t E[g(\epsilon_1 + \delta) - g(\epsilon_1)],$$

where A_t is a positive number depending on the horizon. For the same reason, we get:

$$\begin{aligned} V[Y_t(\delta) - Y_t] &= E[Y_t(\delta) - Y_t]^2 - \left(E[Y_t(\delta) - Y_t]\right)^2 \\ &= B_t E[g(\epsilon_1 + \delta) - g(\epsilon_1)]^2 - A_t^2 \left(E[g(\epsilon_1 + \delta) - g(\epsilon_1)]\right)^2 . \end{aligned}$$

The response function depends on both the magnitude of the shock and the horizon. To clarify the dependence with respect to the shock size, we reproduce in Figures 3.8-3.9 the Figures 3.4-3.5, with δ measured on the x-axis. The response function is convex for the mean deviation with a stronger convexity associated with negative shocks. In particular, the properties of symmetry and linearity of linear impulse responses are not satisfied. The variance function displays an asymmetric effect of positive and negative shocks (related to the so-called leverage effect discussed in the financial literature).

An impulse response analysis based only on the differences $Y_t(\delta)$ - Y_t can be misleading since it doesn't convey information about the shock effect to the transformed series $g(Y_t)$, such as Y_t^2 for volatility analysis, or $1_{Y_t > y}$, for extreme risk

FIGURE 3.8
Mean Deviation from Benchmark

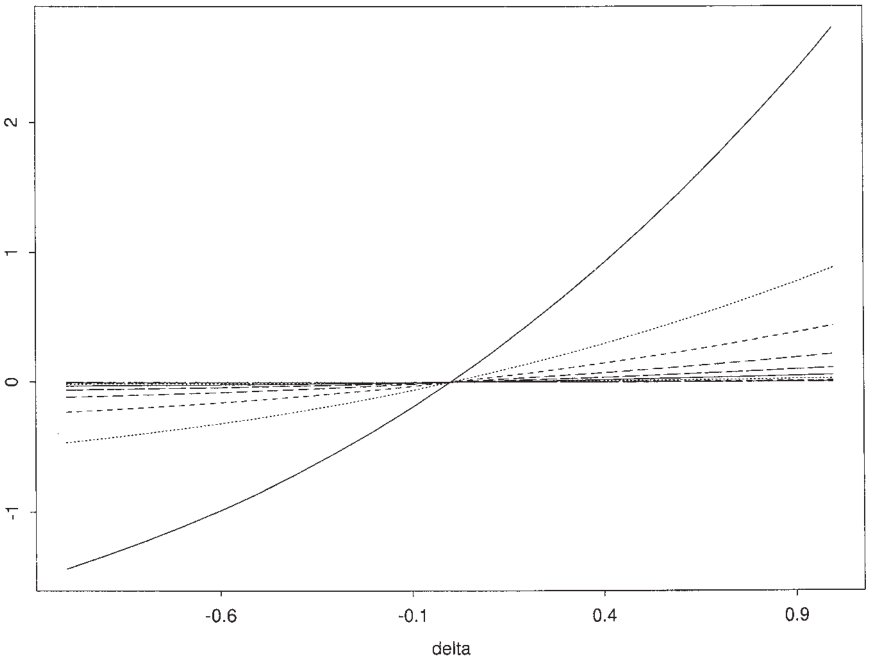
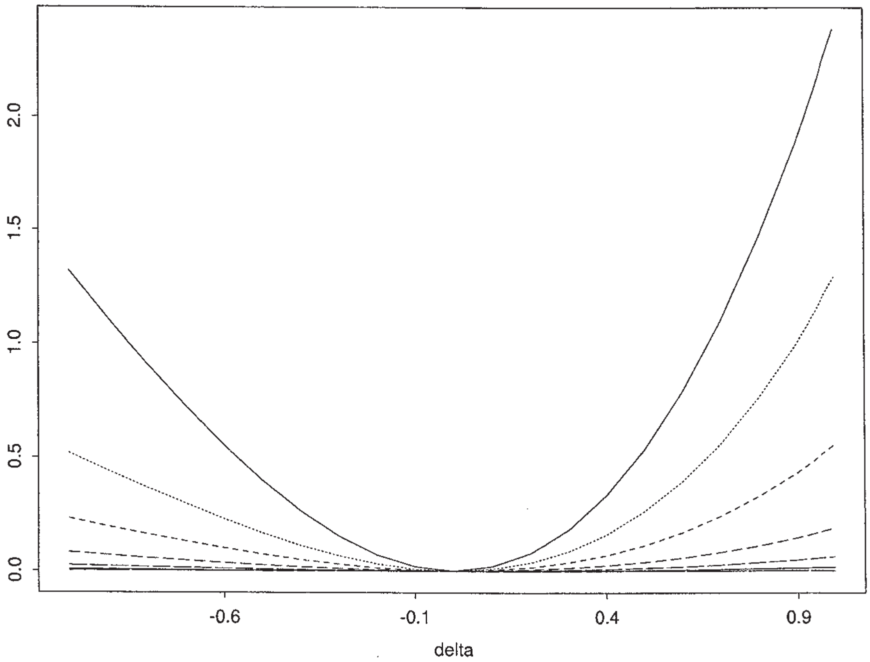
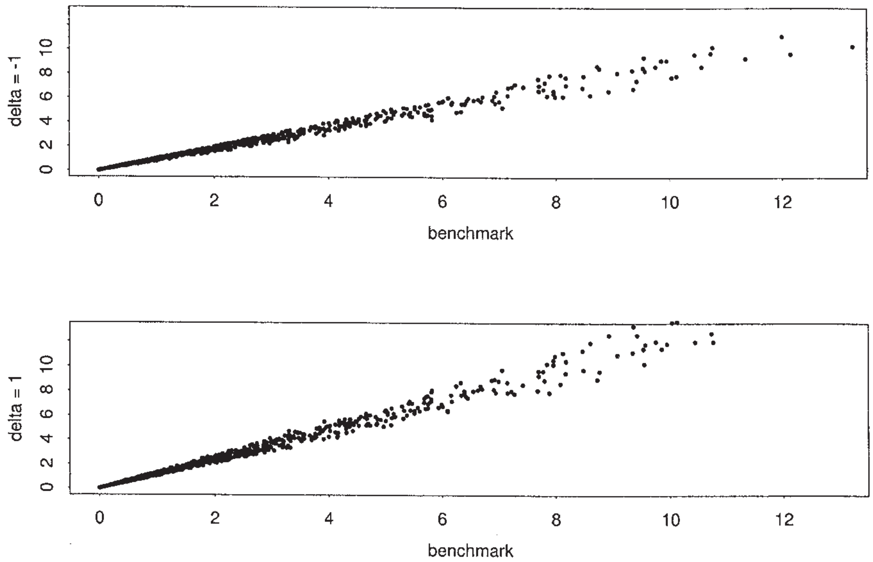


FIGURE 3.9
Variance of the Deviation from Benchmark



analysis, for example. It is more insightful to consider the joint bivariate distribution of $Y_t(\delta)$, Y_t and examine how it depends on the magnitude of the shock and on the horizon. The corresponding scatterplots are given in Figure 3.10 for $\delta = +1,-1$, and horizon 3.

FIGURE 3.10
Scatterplot at Horizon 3



We find that the joint distribution of $[g(Y_t), g(Y_t^s)]$ can be easily derived from the joint distribution of (Y_t, Y_t^s), while the distribution of $(Y_t - Y_t^s)$ cannot be used to derive the distribution of $(g(Y_t^s) - g(Y_t))$. This explains why the joint distribution provides an appropriate representation for impulse responses.

ii) **Model with a linear autoregressive factor**

Let us introduce a Gaussian AR(1) model:

(3.11) $$Z_t = \rho Z_{t-1} + \epsilon_{t,} \quad t \text{ varying},$$

and the process of interest defined by:

(3.12) $$Y_t = a\left(\epsilon_{t,}, Z_t\right), \quad t \text{ varying},$$

where a is a given function. This process is generated by two underlying factors ϵ_t, Z_t based on the same Gaussian white noise. For a transitory shock δ that hits the noise process at date 1, we get:

$$Y_t(\delta) = a(\epsilon_t, Z_t + \rho^{t-1}\delta), \quad t \geq 2,$$

and the joint distribution of $[Y_t, Y_t(\delta)]$ can easily be deduced from the joint Gaussian distribution of (ϵ_t, Z_t). The effect of the shock depends on the nonlinear transformation *a.*

FIGURE 3.11
Scatterplot a = epsilon × Z

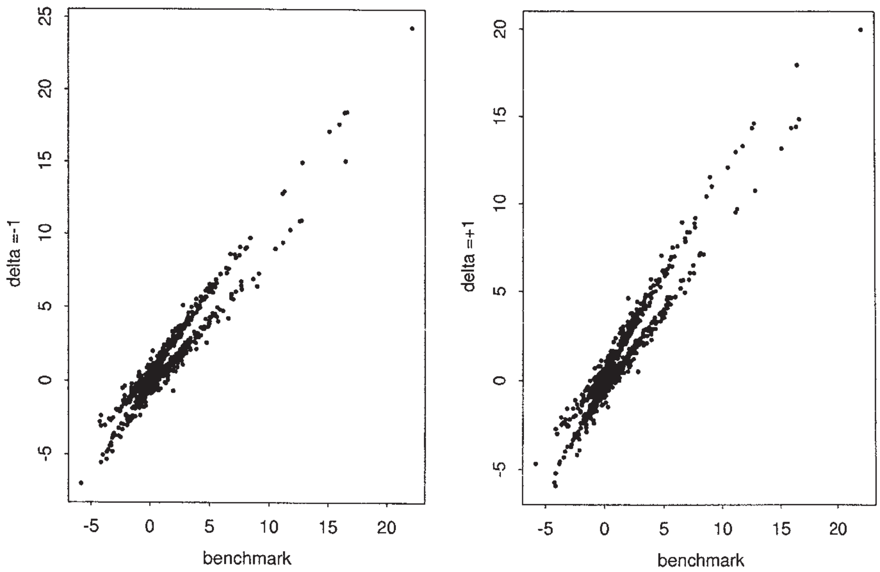
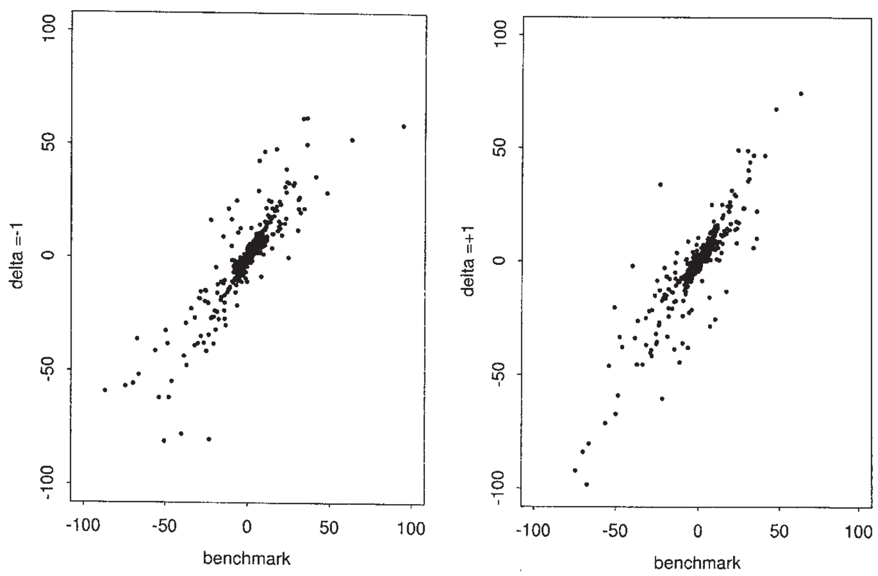


FIGURE 3.12
Scatterplot a = Z / epsilon]



Let us first consider the function:

$$a(\epsilon_t, Z_t) = sign(\epsilon_t)\exp Z_t. \quad (3.13)$$

We see that: $Y_t(\delta) = Y_t \exp(\rho^{t-1}\delta)$, and the joint distribution of $[Y_t, Y_t(\delta)]$ is degenerate so that its support is a line passing through the origin. The shock has no effect on the sign of Y_t, whereas it has a multiplicative effect on its absolute value.

Less extreme examples corresponding to functions: $a(\epsilon_t, Z_t) = \epsilon_t Z_t$ and $a(\epsilon_t, Z_t) = Z_t / \epsilon_t$, respectively, are illustrated in Figures 3.11-3.12. The values of the parameters are: $\rho = 0.9$, $\delta = +1,-1$, and the horizon is $t = 4$.

These examples show that, in some cases, the joint distributions of $[Y_t, Y_t(\delta)]$ have complicated forms which are poorly described by comparing only the expectations or the variances. Therefore a comprehensive impulse response analysis may eventually require more detailed summary statistics.

4 Value-at-Risk for a dynamic financial strategy

The specification and estimation of a dynamic model for economic or financial series are often preliminary procedures in the developing of market trading, portfolio hedging and other dynamic strategies. In such a framework, the decision makers are more interested in the consequences of shocks to the outcomes of a dynamic strategy rather than in their effects on the asset return series itself. Even for time series with simple dynamics, the choice of a strategy is often determined by a solution to either a myopic, or an intertemporal optimization problem, which is always nonlinear. However, impulse responses to nonlinear solutions don't admit closed form expressions. To circumvent this difficulty, it is possible to linearize the outcomes by replacing the criterion function to be optimized by a quadratic criterion function. Such an approach would yield however a biased outcome and inaccurate impulse responses. Therefore, it is preferable to approximate directly the nonlinear impulse responses by simulations. In this section, we discuss the structural interpretation of impulse response functions applied to the Value-at-Risk [VaR] employed in Finance to measure and control portfolio risk. In the first subsection, we review the standard definition of the VaR, and extend this definition to a dynamic strategy of risk assessment in the second subsection.

4.1 Definition of the VaR

Let us consider at date T a portfolio that contains the quantities $a_{0,T}$ and a_T of a riskfree asset and various risky assets, respectively. We denote by $y_{0,T+h}, y_{T+h}$, $h = 0,1,...,H$, the future values of the primitive assets. According to the regulation and monitoring rules (see, Gourieroux, Jasiak [2002]), the VaR is computed for portfolios with allocations constrained to remain fixed in time (the so-called crystallization of the portfolio). The future portfolio values are:

$$W_{T+h=}a_{0,T}\, y_{0,T+h} + a'_T\, y_{T+h},\ h = 0,1,...,H. \tag{4.1}$$

These values are random conditional on the information available at time *T*.

The Value-at-Risk of this portfolio evaluated at time *T*, for horizon *H* and critical level $\alpha \in [0,1]$ is the quantity VaR [T, H, α] defined by:

$$P_T\left[W_{T+H} > -VaR\left[T,H,\alpha\right]\right] = 1-\alpha, \tag{4.2}$$

where P_T is the conditional distribution of future prices. The Value-at-Risk is function of the available information $\underline{Y_T}$ corresponding to prices prior to *T*. The VaR may be used to determine the minimum capital reserve (MCR) that needs to be kept by a bank to ensure that the total wealth, including the portfolio value and the capital requirement, remains positive with a sufficiently large probability. More precisely, let us assume that the MCR can be invested at a rate providing a zero coupon price *B*[*T,H*] for horizon *H*. Consequently, the VaR determined capital requirement would be:

$$MCR\left[T,H,\alpha\right] = \frac{VaR\left[T,H,\alpha\right]}{B\left[T,H\right]}. \tag{4.3}$$

In practice risk is measured ex-post, although it would be optimal to do it ex-ante. The reason for this is the belief that the conditional distribution P_T can be well approximated by the historical distribution estimated from recent data (see e.g. MORGAN [1994]). However, when a dynamic model of asset prices is available, an ex-ante computation of the VaR can be performed by simulation and will provide a different result than the misspecified ex-post (historical) approach, which implicitly assumes i.i.d. returns. For instance, let us suppose that asset prices follow a (nonlinear) AR(1) model:

$$y_t = g\left(y_{t-1}, \epsilon_t\right),\ t \text{ varying}, \tag{4.4}$$

where *g* is a given function, and $y_{0,T+h} = 1,\ \forall h = 1,...,H.$ We can simulate the future risky asset prices given the current value y_T by:

$$y^s_{T+h} = g\left(y^s_{T+h-1}, \epsilon^s_{T+h}\right),\ h = 1,...,H,\ s = 1,...,S, \tag{4.5}$$

where $y^s_T = y_T$, and we deduce the simulated future values of the portfolio as:

$$W^s_{T+h} = W_T + a'_T(y^s_{T+h} - y_T),\ h = 1,...,H,\ s = 1,...,S. \tag{4.6}$$

Then, the VaR can be computed from the empirical distribution of these simulated values $W^s_{T+h}, s = 1,...,S$.

Let us now introduce a shock δ to the innovation at date T+1. Similar computations can be performed after replacing ϵ^s_{T+1} by $\epsilon^s_{T+1} + \delta$, to obtain the distribution of the future portfolio values $W^s_{T+h}(\delta)$, say, along with the associated Value-at-Risk: VaR [T, H, α; δ]. This will allow us to study the sensitivity of the VaR and of the MCR to a transitory shock δ.

4.2 **Extension to dynamic strategies**

The assumption of portfolio crystallization, that is, of fixed portfolio allocation, has been imposed by the regulators for its simplicity, at the sake of optimality. Indeed, investors regularly update portfolio allocations to take advantage of price movements. A typical example is a hedging portfolio for a European call. The portfolio is often updated at regular dates $T+h$, $h = 1,\ldots,H$ (say), with allocations determined by the deltas of the Black-Scholes formula. The idea is to reduce the risk by implementing nonlinear updating strategies. Even though the price evolution in the Black-Scholes model obeys linear dynamics, the deltas and the future portfolio value are complicated nonlinear functions of prices.

Let us still denote the price of the riskfree asset by $y_{0,T+h} = 1$, $h = 1,...,H$. The future values of a self-financed portfolio are:

$$W_{T+H}\left[a(.)\right] = W_T + \sum_{h=1}^{H} a_h(\underline{y_{T+h-1}})\left[y_{T+h} - y_{T+h-1}\right], \quad (4.7)$$

where $a_h(\underline{y_{T+h-1}})$ are the path dependent allocations in the risky assets considered at the h^{th} updating.

For a given dynamic strategy $a(.) = [a_h(.), h = 1,\ldots,H]$, we can compute, like in the previous subsection, the simulated future portfolio values:

$$W^s_{T+H}\left[a(.)\right] = W_T + \sum_{h=1}^{H} a_h(\underline{y^s_{T+h-1}})\left[y^s_{T+h} - y^s_{T+h-1}\right], \quad (4.8)$$

and find the VaR's under a transitory shock and without it. These Values-at-Risk will depend on the selected dynamic strategy. Let us denote them by:

$$VaR\ [T, H, \alpha; a(.)] \text{ and } VaR\ [T, H, \alpha; a(.); \delta].$$

They can be compared to the Values-at-Risk evaluated for a portfolio with fixed allocations $a_h(.) = a_T, \forall h$:

$$VaR\ [T, H, \alpha] \text{ and } VaR\ [T, H, \alpha; \delta],$$

using the notation of Subsection 6.1.

4.3 **Application**

As an illustration, let us consider porfolios invested in one month T-Bill (the risk-free asset) and the NYSE market index (the risky asset). These portfolios may be updated every week. The price of the riskfree asset is set to $y_{0,t} = 1$, whereas the net price of the risky asset is denoted by y_t. The dynamics of the associated net returns r_t has been estimated on weekly data for the period of July 1962 - December 1985 using a GARCH-M model (Chou [1988]). The estimated model is:

(4.9)
$$\begin{aligned} r_t &= \gamma \quad h_{t-1}^2 + h_t \epsilon_t \\ &= \underset{(3.28)}{4.56} \; h_{t-1}^2 + h_t \epsilon_t, \end{aligned}$$

where:

(4.10)
$$\begin{aligned} h_t^2 &= \alpha_0 + \alpha \quad h_{t-1}^2 \epsilon_{t-1}^2 + \beta \quad h_{t-1}^2 \\ &= \underset{(2.83)}{0.99 * 10^{-4}} + \underset{(7.15)}{0.151} \; h_{t-1}^2 \epsilon_{t-1}^2 + \underset{(38.64)}{0.834} \; h_{t-1}^2, \end{aligned}$$

and $\epsilon_t \sim IIN(0,1)$. The t-ratios are given in brackets. Note that lagged volatility has been introduced into the conditional mean equation to avoid a time independent risk premium.

Let us consider arbitrage portfolios with zero initial investment $W_0 = 0$ and a horizon $H = 52$ weeks corresponding to one year. The first portfolio is static with a mean-variance efficient allocation (Markowitz [1952]) computed at the initial date and associated with risk aversion coefficient $A = 1$[5]. Its value at the terminal date is:

(4.11)
$$W_H = \frac{\gamma h_{-1}^2}{h_0^2 y_0} \left(y_H - y_0 \right).$$

The second portfolio is regularly updated every week using a time dependent mean-variance allocation and the same risk aversion coefficient. Its terminal value is:

(4.12)
$$W_H(a) = \sum_{j=1}^{H} \frac{\gamma h_{j-1}^2}{h_j^2 y_{j-1}} \left(y_j - y_{j-1} \right).$$

Thus, the complicated dynamic nonlinear effects are due to both the GARCH-M estimated dynamics of the net returns and the nonlinear relationship between the portfolio value of the self-financed portfolio and the return history.

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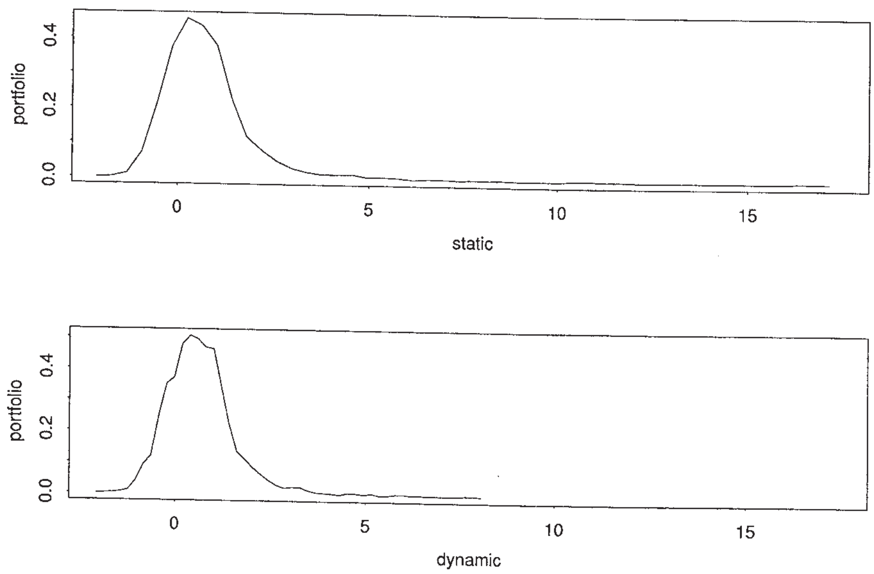
5. Due to zero investment, the terminal values of the static and dynamic portfolios are simply multiplied by 1/A in the case of a risk aversion coefficient different from 1. This does not modify the distribution patterns.

Even though a sequence of myopic reallocations is not optimal from the intertemporal optimization perspectives, such a dynamic strategy is expected to outperform the static approach. However, it is not clear whether the dynamic strategy is better with respect to both the expected portfolio value and its volatility. Moreover, one could fear that this more sophisticated strategy is less robust and more sensitive to external shocks. To examine these issues, let us fix the initial value h_{-1}^2 of the volatility equal to its expectation estimated from the residuals of the last iteration step in (4.10), and the value y_0 equal to the last value from the sample. We show in Figure 4.1 the p.d.f. of the terminal portfolio values for the static and dynamic portfolio management strategies at horizon $H = 52$.

FIGURE 4.1
Distribution of Terminal Values

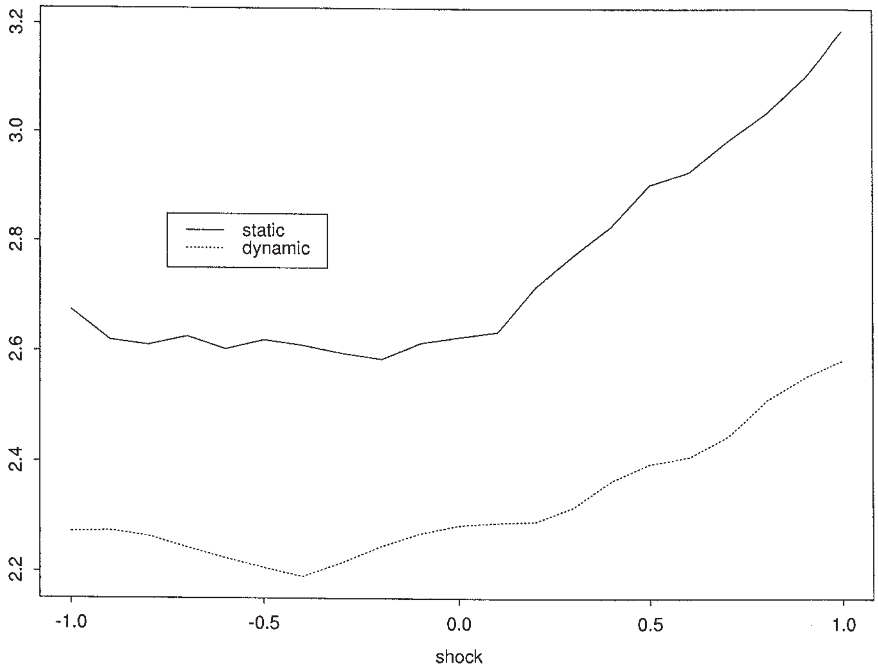


The means of the distributions are quite close whereas the tail of the distribution is heavier in the case of a static portfolio. Therefore, the choice of an appropriate dynamic strategy may reduce the risk[6]. This effect is confirmed when we study the shock sensitivity of the VaR. Below we display the VaR as a function of a transitory shock to the innovation at the initial date for the observed path starting in July 1962 and a shock of size between -1 and 1.

We find that the required amount of reserve is not only lower under the dynamic management for all shock sizes, but is also less sensitive to innovation shocks. Indeed, the slope of the VaR is weaker for the dynamic strategy. We also observe asymmetric reactions of the VaR to positive and negative shocks. At first sight, this result seems surprising. Indeed, it is commonly believed that major risks in portfolio management are related to the trader's behavior, or more precisely to the way the portfolio allocation updating is caried out, and not to asset price uncertainty.

6. Even if the BLACK-SCHOLES deltas don't provide the optimal hedging allocation for GARCH-M return dynamics.

FIGURE 4.2
VaR for Static and Dynamic Strategies



The example above shows that adequate allocation updating will diminish the risk and enhance the portfolio robustness to shocks.

Our analysis could have been pursued by considering the joint distribution of the two Values-at-Risk over admissible future paths. In particular, it could have been interesting to report the probability of the VaR under static portfolio management be less than the VaR under dynamic management. Such a computation would require the marginal distribution of $Y_t(\delta)$ only. However, the joint distribution of the perturbed and unperturbed paths is needed for stress testing as suggested by the regulators. The regulator is interested in considering some given disturbance δ and fixing the required capital as a function of the supremum of the perturbed and unperturbed VaR. The determination of the distribution of $Max[VaR(T, H, \alpha), VaR(T, H, \alpha, \delta)]$, requires joint simulations of the perturbed and unperturbed price paths.

5 Conclusions

This paper introduced the notion of nonlinear innovations for time series models which provides a unifying framework for the analysis of processes and their nonlinear transformations. The nonlinear impulse response analysis proposed in this paper covers both permanent and transitory shocks, and accounts for the asymmetric effects of shocks, shock nonlinearity and path dependency typical for nonlinear processes. It reveals that the joint distribution of the perturbed and unperturbed paths provides a valid and comprehensive representation of impulse responses. The empirical part of the paper shows the impulse response analysis in the Value-at-Risk evaluation under a static and a dynamic portfolio management strategy.

References

ABRAMOWITZ M. and STEGUN T. (1964). – « Handbook of Mathematical Functions », *National Bureau of Standards*, Applied Mathematical Series.

BLANCHARD O. and QUAH D. (1989). – « The Dynamic Effects of Aggregate Demand and Aggregate Supply Disturbances », *American Economic Review*, 79, pp. 655-73.

BOUGEROL P. and PICARD N. (1992). – « Stationarity of GARCH Processes », *Journal of Econometrics*, 52, pp. 115-27.

CHOU R.Y. (1988). – « Volatility Persistence and Stock Valuations: Some Empirical Evidence Using GARCH », *Journal of Applied Econometrics*, 3, pp. 279-94.

ENGLE R. (1982). – « Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of UK Inflation », *Econometrica*, 50, pp. 987-1008.

ENGLE R. and RUSSELL J.R. (1998). – « The Autoregressive Conditional Duration Model », *Econometrica*, 66, pp. 1127-63.

GALLANT A., ROSSI P. and TAUCHEN G. (1993). – « Nonlinear Dynamic Structures », *Econometrica*, 61, pp. 871-908.

GOURIEROUX C. and JASIAK J. (2002). – « Financial Econometrics », *Princeton University Press*.

GOURIEROUX C. and MONFORT A. (1997). – « Time Series and Dynamic Models », *Cambridge University Press*.

GRANGER C. (1995). – « Modelling Nonlinear Relationships between Extended Memory Variables », *Econometrica*, 63, pp. 265-79.

GRANGER C. and NEWBOLD P. (1976). – « Forecasting Transformed Series », *Journal of the Royal Statistical Society*, B, 38, pp. 189-203.

GRANGER C. and TERASVIRTA T. (1993). – « Modelling Nonlinear Economic Relationships », *Oxford University Press*.

INOUE A. (1997). – « A Conditional Goodness of Fit Test in Time Series », *manuscript, Department of Economics*, University of Pennsylvania.

KILIAN L. (1998). – « Small Sample Confidence Intervals for Impulse Response Functions », *Review of Economics and Statistics*, 80, pp. 218-30.

KLEIN L. (1998). – « Small Sample Confidence Intervals for Impulse Response Functions », *Review of Economics and Statistics*, 80, pp. 218-30.

KOOP G., PESARAN H. and POTTER S. (1996). – « Impulse Response Analysis in Nonlinear Multivariate Models », *Journal of Econometrics*, 74, pp. 119-47.

LUTKEPOHL H. and SAIKKONEN P. (1997). – « Impulse Response Analysis in Infinite Order Cointegrated Vector Autoregressive Processes », *Journal of Econometrics*, 81, pp. 127-57.

MARKOWITZ H. (1952). – « Portfolio Selection », *Journal of Finance*, VIII, pp. 77-91.

MORGAN J.P. (1994). « Risk Metrics: Technical Document », *2nd edition*, New York.

NELSON D. (1990). – « Stationarity and Persistence in the GARCH(1,1) Model », *Econometric Theory*, 6, pp. 318-34.

NISIO M. (1960). – « On Polynomial Approximation for Strictly Stationary Processes », *Journal of Math. Soc. Jpn*, 12, pp. 207-76.

OSELEDEC V. (1968). – « A Multiplicative Ergodic Theorem: Liapunov Characteristic Numbers for Dynamical Systems », *Trans. Moscow Math. Soc.*, 19, pp. 197-231.

PHILLIPS P.C.R. (1998). – « Impulse Responses and Forecast Error Variance Asymptotics in Nonstationary VAR's », *Journal of Econometrics*, 83, pp. 21-56.

PRIESTLEY H. (1988). – « Non-Linear and Non-Stationary Time Series Analysis », *Academic Press*.

SIMS C. (1972). – « Money, Income and Causality », *American Economic Review*, 62, pp. 540-52.

SIMS C. and ZHA T. (1999). – « Error Bands for Impulse Responses », *Econometrica*, 67, pp. 1113-55.

TONG H. (1990). – « Nonlinear Time Series: A Dynamic System Approach », *Oxford University Press*.

VOLTERRA V. (1930). – « Theory of Functionals », *Blackie*, London.

VOLTERRA V. (1959). – « Theory of Functionals and of Integro-Differential Equations », *Dover*, New-York.

WHITTLE P. (1963). – « Prediction and Regulation », *English University Press*, London.

WIENER N. (1958). – « Non-Linear Problems in Random Theory », *MIT Press*, Cambridge, Massachussetts.

YAO Q. and TONG H. (1994). – « Quantifying the Influence of Initial Values on Nonlinear Prediction », *Journal of the Royal Statistical Society*, B 56, pp. 701-25.