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Present Value Models with Feedback: Dynamic Properties of Alternative RE Equilibria

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ABSTRACT. – This paper analyzes the dynamic features displayed by alternative rational expectations equilibria in the context of a simple present value model with feed-back. We show how these features change for small perturbations of the parameters characterizing the forcing variable process. Moreover, we derive some implications of the analysis for both econometric practice and econometric policy evaluation. In particular, our analysis illustrates scenarios where we cannot rule out the possibility that an economy may switch from a 'Lucas proof' equilibrium to an equilibrium which is not immune to the *Lucas critique* when some economic policies are implemented.

Modèle à valeur présente escomptée avec *FeedBack* : propriétés dynamiques des équilibres à attentes rationnelles

RÉSUMÉ. – Cet article analyse les propriétés dynamiques de différents équilibres à anticipations rationnelles dans un modèle simple à valeur présente escomptée avec *feed-back*. Nous montrons que ces propriétés se modifient suite à des perturbations minimes des paramètres qui décrivent le processus suivi par la variable exogène du modèle. Ensuite, nous en dérivons des implications pour l'estimation de ce type de modèle, ainsi que pour l'évaluation économétrique de diverses politiques économiques. En particulier, notre analyse met en évidence des scénarios qui ne nous permettent pas d'exclure la possibilité qu'une économie passe d'un équilibre invariant à la *critique de Lucas* à un autre équilibre qui lui est affecté par cette critique.

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1 Introduction

Present value (PV) relations are considered to be among the simplest dynamic stochastic models used in economics (for instance, CAMPBELL and SHILLER [1987]). Examples of PV relations can be found in models of the stock market, models of the term structure of interest rates, models of aggregate consumption, monetary models of exchange rates and models of hyperinflation. The analysis of PV models has long been characterized by the assumption that the forcing variable follows an exogenous process. To our knowledge, TIMMERMANN [1994] is the first paper that analyzes the implications of having a feedback mechanism from the endogenous variable to the forcing variable in the context of a general linear PV model under rational expectations (RE).[1] From the analysis of PV relations with feedback, *Timmermann* shows that the presence of feedback leads to the existence of multiple *bubble-free* RE equilibria. *Timmermann* then argues that switches between RE equilibria may be induced by variations in the parameters characterizing the forcing variable process.[2] However, *Timmermann* neither explains which type of mechanisms may lead the economy to switch between RE equilibria nor discusses the implications of multiple equilibria for both econometric practice and econometric policy evaluation.

The aim of this paper is twofold. First, we extend *Timmermann*'s analysis in two directions. On the one hand, we study the dynamic properties of alternative RE equilibria. On the other hand, we analyze how the dynamic features of alternative equilibria may change due to small perturbations in the parameters describing the forcing variable process. This characterization allows us to describe mechanisms explaining switches between RE equilibria (that is, *switching equilibria*). Second, we discuss several relevant issues for both econometric practice and econometric policy evaluation arising in the presence of multiple equilibria combined with the likelihood of switching equilibria. In particular, conditional on several selection criteria widely used in the literature, our analysis characterizes scenarios where we cannot rule out the possibility that the economy may switch from a '*Lucas* proof' equilibrium to an equilibrium which is not immune to the *Lucas critique* when some economic policies are implemented.

We consider five equilibrium selection criteria as a way of pointing out some *desirable* features of the equilibria analyzed and to illustrate how alternative RE equilibria differ in several dimensions. We first consider that the equilibrium solutions must be real, rather than complex, since a complex

1. The presence of a feedback mechanism implies that the forcing variable is also an endogenous variable. The paper by EVANS and HONKAPOHJA [1986] is an early paper in considering that the forcing variable is also endogenous, but they start from a reduced form model with a single endogenous variable.

2. Another important conclusion obtained from *Timmermann*'s analysis is that the feedback mechanism rules out the presence of speculative bubbles in which the endogenous variable grows asymptotically at a larger rate than the forcing variable. The reason is that any bubble in the endogenous variable process that grows asymptotically will feed into the forcing variable process due to the feedback mechanism, such that a growing discrepancy between the endogenous and forcing variable is not possible.

solution is not economically sensible (McCallum [1983]). We refer to this criterion as the *definition* criterion. Second, we consider the *stationary* criterion that selects from among alternative RE equilibria those in which the endogenous variable is stationary. The use of this selection criterion has a long standing tradition (for instance, Sargent and Wallace [1975] and Phelps and Taylor [1977]). Third, the *minimum variance* criterion proposed by Taylor [1977]. Fourth, the *minimal state variable* criterion suggested by McCallum [1983, 1999]. Finally, the immunity to the *Lucas critique* criterion proposed by Farmer [1991].

The paper is organized as follows. Section 2 introduces a simple PV model with feedback and obtains the alternative RE equilibrium solutions. Moreover, this section reviews the features of alternative equilibria in a PV model without feedback. Section 3 characterizes the dynamic properties displayed by alternative RE equilibria in the space of the parameters describing the forcing variable process. Section 4 discusses some implications of the analysis which are relevant for both econometric practice and econometric policy evaluation.

2 The Present Value Model

For illustration purposes, we focus our attention on a simple PV model with feedback. Consider the following PV model:

$$y_t = c + (1-\delta)\, z_t + \delta E_t y_{t+1} + u_t, \tag{1}$$

where $0 < \delta < 1$ and c are parameters, u_t is an i.i.d. random measurement error term with mean zero and variance σ_u^2 and E_t denotes the conditional expectation operator given the information set, I_t, available to the economic agents at the beginning of time t. I_t includes current and past values of all random variables of the model. y_t is the endogenous variable and z_t is the forcing variable which is assumed to be characterized by the following process:

$$z_t = \rho_0 + \rho_1 y_{t-1} + \rho_2 z_{t-1} + v_t, \tag{2}$$

where ρ_0 is a constant, ρ_1 and ρ_2 are both included in the interval $[0,1]$, and v_t is an i.i.d. random variable with mean zero and variance σ_v^2. v_t is included in I_t since z_t is also included. Moreover, equation (2) allows for the presence of a positive feedback from the endogenous variable to the forcing variable. As pointed out by Timmermann [1994], there are several ways of rationalizing this feedback depending on the PV relation studied. For instance, a positive feedback relation from consumption to labor income can be viewed as a Keynesian multiplier in a dynamic context. In a hyperinflationary model, a positive feedback from prices to money supply may reflect a *passive* monetary policy where seigniorage is used finance a real fiscal deficit. In models of the term structure of interest rates, a positive feedback relation from a long-term interest rate to a short-term interest rate can be rationalized by aggregation of asymmetric information, thus a long-term rate aggregates private information that can be used to forecast the evolution of a short-term rate.

Models of the term structure of the interest rate, models of aggregate consumption, and models of hyperinflation can be cast on the same structure given by equations (1) and (2).[3]

Equations (1) and (2) form a bivariate system of difference equations. Using the undetermined coefficient method (MUTH [1961], MCCALLUM [1983] among others) we begin by writing y_t as a linear function of u_t, z_t plus a constant,

$$y_t = \pi_1 + \pi_2 z_t + \pi_3 u_t. \tag{3}$$

For appropriate real values of π_1 and π_2, the expectational variable $E_t y_{t+1}$ will then be given by:

$$E_t y_{t+1} = \pi_1 + \pi_2 E_t z_{t+1} = (\pi_1 + \pi_2 \rho_0) + \pi_2 \rho_1 y_t + \pi_2 \rho_2 z_t. \tag{4}$$

To evaluate the π's, we substitute (2), (3) and (4) into (1), which gives:

$$\begin{aligned}\pi_1 + \pi_2 z_t + \pi_3 u_t &= c + \delta \pi_1 (1 + \pi_2 \rho_1) + \delta \rho_0 \pi_2 \\ &\quad + [\delta \pi_2 (\pi_2 \rho_1 + \rho_2) + (1 - \delta)] z_t + [1 + \delta \pi_3 \pi_2 \rho_1] u_t.\end{aligned}$$

This equation implies identities in the constant term, z_t and u_t as follows:

$$\begin{aligned}\pi_1 &= c + \delta \pi_1 (1 + \pi_2 \rho_1) + \delta \rho_0 \pi_2, \\ \pi_2 &= \delta \pi_2 (\pi_2 \rho_1 + \rho_2) + (1 - \delta), \\ \pi_3 &= 1 + \delta \pi_3 \pi_2 \rho_1.\end{aligned} \tag{5}$$

After some algebra, we can show that there are two solutions to the system of equations (5),

$$\pi^1 \equiv \left(\pi_1^1, \pi_2^1, \pi_3^1\right) = [\tau_1, \alpha_1, \varphi_1], \tag{6}$$

$$\pi^2 \equiv \left(\pi_1^2, \pi_2^2, \pi_3^2\right) = [\tau_2, \alpha_2, \varphi_2], \tag{7}$$

where:

$$\begin{aligned}\alpha_1 &= \tfrac{1}{2\delta\rho_1}\left[1 - \delta\rho_2 + \sqrt{(1 - \delta\rho_2)^2 - 4\rho_1\delta(1 - \delta)}\right], \\ \alpha_2 &= \tfrac{1}{2\delta\rho_1}\left[1 - \delta\rho_2 - \sqrt{(1 - \delta\rho_2)^2 - 4\rho_1\delta(1 - \delta)}\right], \\ \tau_i &= \tfrac{c + \alpha_i \rho_0 \delta}{1 - \delta(1 + \alpha_i \rho_1)} \qquad i = 1,2, \\ \varphi_i &= \tfrac{1}{1 - \alpha_i \rho_1 \delta}, \qquad i = 1,2.\end{aligned}$$

The coefficients α_1 and α_2 are the roots of the second-order characteristic polynomial in π_2 given by the second equation in system (5).

3. The PV model for stock prices can also be cast in the same structure but taking into account that the current value of dividends (that is, z_t, the forcing variable in this model), whose realization occurs during the period, does not belong to current agent's information set I_t. This feature of the PV model of stock prices induces a rather different dynamic picture of the equilibria from that obtained from the PV model studied in this paper. The dynamic properties of the alternative RE equilibria arising in the PV model for stock prices and the presence of switching equilibria in the US stock market are analyzed in a companion paper by GUTIÉRREZ and VÁZQUEZ [2000].

In addition to RE solutions (6) and (7), the PV model, equation (1), exhibits another RE equilibrium solution. This alternative solution is obtained by following the backward approach for solving linear RE models (see BROZE and SZAFARZ [1991, ch.2]). This approach starts by using the fact that the RE of the endogenous variable at period t is given by:

$$E_t y_{t+1} = y_{t+1} - \varepsilon_{t+1}, \tag{8}$$

where ε_{t+1} (the rational prediction error) is an arbitrary martingale difference with respect to agents' information set at period t, I_t. By using (8) and rearranging, the PV model (1) can be written as:

$$y_t = -\delta^{-1}c + \delta^{-1}y_{t-1} + \left(1 - \delta^{-1}\right) z_{t-1} - \delta^{-1}u_{t-1} + \varepsilon_t. \tag{9}$$

We refer to equilibrium solution (9) as the backward solution to the PV model. Equilibrium solutions (6) and (7) are called fundamental solutions in the sense that both equilibrium solutions are only linear functions of a minimal set of state variables: z_t and u_t. Notice that the fundamental solutions do not include variables such as y_{t-1},u_{t-1} and ε_t which enter in the backward solution. From now on we refer to solutions (6) and (7) as the α_1-fundamental and α_2-fundamental solutions, respectively.

At this point of the analysis, it is worth making some remarks on the set of RE solutions (equations (6), (7) and (9)):

Remark 1. *The backward solution (9) is a general RE solution for the PV model because it does not depend on the process followed by the forcing variable. Moreover, any particular solution of the PV model (1) (for instance, the fundamental solutions) satisfies (9).*[4]

Remark 2. *In spite of the previous remark, it must be clear that the time series obtained from the three alternative RE solutions display very different dynamic properties. In particular, as shown below in Propositions 3 and 4, the variance of the endogenous variable is rather different depending on which RE equilibrium characterizes the dynamics.*

Remark 3. *Fundamental solutions (6) and (7) satisfy condition (8) even though it was not imposed to derive them. That is, α_1-fundamental and α_2-fundamental solutions are particular RE solutions with the martingale difference term being characterized by a specific linear function of the measurement error u_t (see footnote 4).*

Remark 4. *Fundamental solutions (6) and (7) only exist when:*[5]

$$(1 - \delta\rho_2)^2 - 4\rho_1\delta(1 - \delta) \geqslant 0, \tag{10}$$

that is, when α_1 and α_2 are both real numbers.

4. It can be shown that the fundamental solutions, (6) and (7), satisfy the linear difference equation (9) with the difference martingale given by:

$$\varepsilon_t = \alpha_i v_t + \varphi_i u_t,$$

for $i = 1,2$, respectively.

5. Following MCCALLUM [1983, p. 146, footnote #9], we implicitly believe that if some of the equilibrium solutions are real and the others are complex, then the latter are irrelevant because they are not economically sensible solutions.

Remark 5. *The backward equilibrium solution is characterized by a linear restriction such that the sum of the coefficients associated with y_{t-1} and z_{t-1} is one. The two fundamental solutions do not satisfy this restriction.*

Before analyzing the properties of alternative RE equilibria arising in a PV model with feedback, we first review a simple PV model without feedback (that is, imposing the restriction $\rho_1 = 0$ in (2)). The latter PV model has been extensively analyzed by FARMER [1991, 1999]. If $\rho_1 = 0$, on the one hand, we can show that the α_1-fundamental solution does not exist and the α_2-fundamental solution is given by:

$$y_t = \frac{c}{1-\delta} + \frac{\rho_0 \delta}{(1-\delta\rho_2)} + \frac{(1-\delta)}{(1-\delta\rho_2)} z_t + u_t. \tag{11}$$

Alternatively, the RE solution (11) is also found by iterating equation (1) forwards and substituting for future values for z_t using the forcing variable process (2) with $\rho_1 = 0$. The important property of solution (11) is that the coefficient associated with z_t depends on ρ_2, which implies that the reduced form of the endogenous variable depends on the forcing variable process. This dependence is known as the *hallmark* of RE models. This fact does not imply that any reduced form associated with an endogenous variable in a forward-looking model depends always on the forcing variable parameters. Appendix C reviews an optimal growth model with partial depreciation of capital to show that in particular the reduced form associated with the *gross real interest rate* does not depend on the parameters characterizing the total factor productivity, which is the forcing variable in the optimal growth model.

On the other hand, if $\rho_1 = 0$ the backward solution (9) is explosive whenever $0 < \delta < 1$. Therefore, if $0 < \delta < 1$, there is only one stable RE equilibrium given by (11) and the stationary criterion is then sufficient to select a unique equilibrium in models without feedback.

FARMER [1991] points out that if $\delta > 1$ the backward solution (9) is stable. Moreover, whenever $\delta > 1$ equilibrium solution (11) does not exist, since iterating equation (1) forwards leads to a non-converging weighted sum of future expected values of z_t. Furthermore, *Farmer* points out two features characterizing the backward solution. First, the parameters describing the backward solution are independent of the parameters of the forcing variable process, which implies that the *Lucas critique* does not hold in a scenario characterized by this model whenever $\delta > 1$. Second, the backward solution represents an infinite number of RE equilibria, indexed by the arbitrary martingale difference ε_t. Thus, the stationary criterion is not sufficient to single out an RE equilibrium when $\delta > 1$. However, the minimum variance criterion selects among the set of backward solutions given by (9) the solution satisfying $\varepsilon_t = 0$ for all t. In short, the issue of multiple stationary equilibria only arises in a PV model without feedback when $\delta > 1$. The following section illustrates the existence of multiple equilibria in a simple PV model with feedback, described by equations (1)-(2), even when $\delta < 1$.

3 A Characterization of the Alternative Equilibria

In the context of a general PV model, TIMMERMANN [1994] shows that the presence of feedback leads to the existence of multiple bubble-free RE equilibria. *Timmermann* argues that switches between RE equilibria may be induced by changes in the forcing variable parameters. However, *Timmermann* does not explain what type of mechanisms may lead to switching equilibria.

The aim of this section is to show how variations in the parameters describing the forcing variable process may change the set of feasible RE equilibria defined according to alternative selection criteria. Thus, we can explain switches between RE equilibria when an RE equilibrium is the only one feasible before a change of the forcing variable parameters according to a particular selection criterion, but it is not longer feasible after the change in those parameters.

Since adopting any particular selection equilibrium criterion is *ad hoc*, we use the selection criteria as a way of pointing out some *desirable* properties of the equilibria analyzed. First, following MCCALLUM [1983], we consider the definition criterion that selects real, rather than complex, equilibrium solutions since the latter are considered irrelevant because they are not economically sensible solutions. Second, the stationary criterion selects among real alternative RE equilibria, those in which the endogenous variable and the forcing variable are stationary. Third, following TAYLOR [1977] we take into account the minimum variance criterion to select among alternative real stationary RE equilibria. Fourth, we consider the minimal state variable criterion suggested by MCCALLUM [1983, 1999]. *McCallum*'s criterion selects among real stationary RE equilibria, the RE equilibria containing a minimal set of state variables with the additional requirement that the solution must be valid for any admissible value of the parameters of the model. Finally, FARMER's [1991] selection criterion suggests that agents experiencing frequent changes in the forcing variable process may find it useful to choose a stable (stationary) self-fulfilling forecast rule independent of parameters describing both economic policy and forcing variable processes. Therefore, the real stationary RE equilibrium supported by this forecast rule is independent of the parameters characterizing the forcing variable process (that is, the equilibrium is immune to the *Lucas critique*).

It is clear from this review of alternative selection criteria that selection criteria are usually built upon other selection criteria previously proposed in the literature. Thus, *Taylor*'s minimum variance criterion, *McCallum*'s minimal set of state variables criterion and *Farmer*'s immunity to the *Lucas critique* criterion select among real stationary RE equilibria, trying to single out a RE equilibrium (or at least trying to narrow down the set of feasible RE equilibria).

McCallum's minimal set of state variables criterion is the criterion most frequently used (either explicitly or implicitly) in modern intertemporal macroeconomics. For instance, several numerical methods used to solve dynamic stochastic general equilibrium models use the minimal set of state variables criterion in order to characterize either policy rules as in log-linear approximation methods (for instance, KING, PLOSSER and REBELO [1988] and UHLIG [1999]), or conditional expectations as in the parameterized expectation algorithm suggested by DEN HAAN and MARCET [1990].

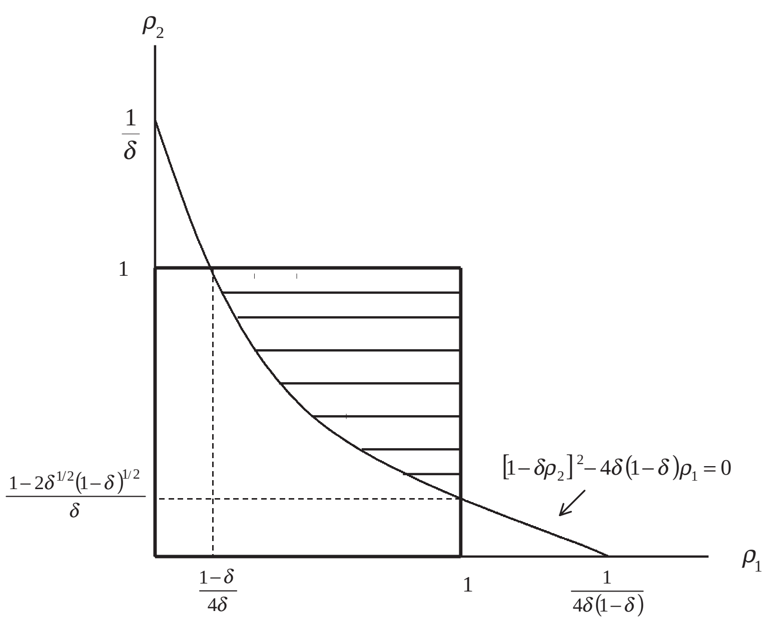
This section firstly shows that the space of combinations of $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$ can be divided in different regions according to the selection criteria considered. The following proposition establishes the region in which only the backward solution exists according to the definition criterion.

PROPOSITION 1. *According to the definition criterion, for all combinations of* $0 < \rho_1 < 1$ *and* $0 < \rho_2 < 1$ *such that inequality* (10) *holds, the three solutions considered exist, otherwise only the backward solution exists. Moreover, the set of values of* ρ_1 *and* ρ_2 *for which only the backward solution exists is empty for* $\delta \leqslant 1/5$. *The set of combinations of* ρ_1 *and* ρ_2 *for which the three solutions exist is not-empty for any* $0 < \delta < 1$.

PROOF: See Appendix A.

Figure 1 summarizes the results stated in Proposition 1. For a given $0 < \delta < 1$, the shaded region displays the combinations of ρ_1 and ρ_2 for which the RE equilibrium is characterized only by the backward equilibrium solution (9) (that is, when inequality (10) is not satisfied). Notice that a value of $\delta = 1/2$, produces the largest region in which only the backward solution exists.

FIGURE 1
Definition Criterion



PROPOSITION 2. *The backward solution* (9) *with* $\varepsilon_t = 0$ *for all t is the single equilibrium solution satisfying Farmer and McCallum criteria.*

PROOF: On the one hand, it is straightforward to see that the backward solution (9) satisfies *Farmer*'s criterion, since this solution is the only one that is independent of the parameters characterizing the forcing variable process. On the other hand, as shown in Proposition 1, the equilibrium solutions (6) and (7) do not exist for admissible values of ρ_1, ρ_2 and δ satisfying $(1-\rho_2\delta)^2 - 4\rho_1\delta(1-\delta) < 0$. Hence, the backward equilibrium solution (9) with $\varepsilon_t = 0$ for all t is the only equilibrium solution pointed out by *McCallum*'s minimal state variable criterion that requires that the solution must be valid for any admissible value of the parameters of the model. This completes the proof.

The following proposition states when the two fundamental solutions are stationary. Moreover, it is shown that the α_2-fundamental solution exhibits a lower variance than the α_1-fundamental solution.

PROPOSITION 3. *Let us assume that* $\mu_i = \rho_2 + \alpha_i\rho_1$ *is a real variable for* $i = 1,2$. *Then fundamental solutions (6) and (7) are stationary if* $\mu_i < 1$ *for* $i = 1,2$, *respectively. If this is the case, the variance of the endogenous variable under the fundamental solutions, (6) and (7), is given by:*

$$\lambda_0^i = \frac{\alpha_i^2}{1-\mu_i^2}\sigma_v^2 + \frac{1+\rho_2^2 - 2\rho_2\mu_i}{1-\mu_i^2}{\varphi_i}^2\sigma_u^2, \tag{12}$$

for $i = 1,2$, *respectively. Furthermore, the variance of the endogenous variable for the* α_2*-fundamental solution, (7), is always lower than for the* α_1*-fundamental solution, (6).*

PROOF: See Appendix A.

Proposition 3 establishes that when the fundamental solutions exist and are stationary, then the α_2-fundamental solution dominates the α_1-fundamental solution according to the minimum variance criterion.

Figure 2 illustrates the regions in which the α_1-fundamental and α_2-fundamental solutions are stationary for the case in which $\delta \geqslant 1/2$. Curve $\mu_1 = \mu_2$ displays combinations of ρ_1 and ρ_2 such that $(1-\delta\rho_2)^2 - 4\rho_1\delta\,(1-\delta) = 0$. Therefore, region B is the area where only the backward solution is defined. The segment connecting the points $(\rho_1,\rho_2) = (\frac{1-\delta}{\delta}, \frac{2\delta-1}{\delta})$ and $(\rho_1,\rho_2) = (1,0)$ displays the pairs (ρ_1,ρ_2) such that $\mu_1 = 1$. Points located to the right of this segment imply combinations of ρ_1 and ρ_2 for which the α_1-fundamental solution is stationary. Otherwise, the α_1-fundamental solution is not stationary. The segment from $(\rho_1,\rho_2) = (0,1)$ to $(\rho_1,\rho_2) = (\frac{1-\delta}{\delta}, \frac{2\delta-1}{\delta})$ displays combinations of ρ_1 and ρ_2 such that $\mu_2 = 1$. Points above this segment result in combinations of ρ_1 and ρ_2 for which the α_2-fundamental solution is not stationary; otherwise, the α_2-fundamental solution is stationary. All these statements about the figures are shown in Appendix B.

FIGURE 2
Stationary Criterion for $\delta \geqslant 1/2$

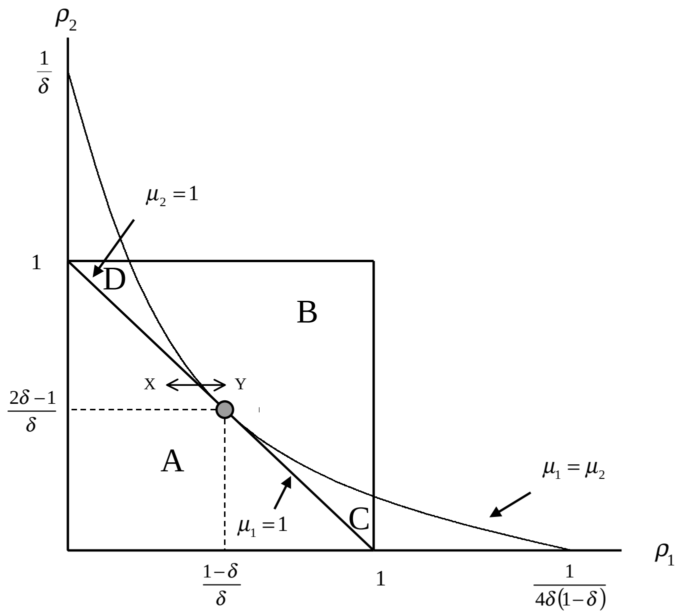
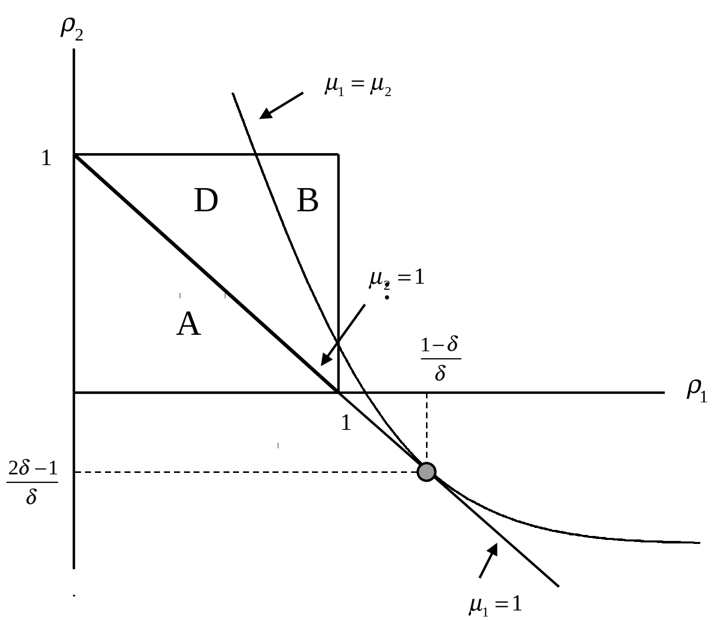


FIGURE 3
Stationary Criterion for $\delta \leqslant 1/2$



Summarizing, in the area in which the two fundamental solutions exist (all regions but B) and for the case in which $\delta \geqslant 1/2$, we can distinguish three different regions (A, C and D) depending upon the stationarity features of these solutions. In particular, the two fundamental solutions are stationary in region C, whereas only the α_2-fundamental solution is stationary in region A. However, region D shows the pairs (ρ_1, ρ_2) for which none of the two fundamental solutions is stationary. As shown in Proposition 3, the α_2-fundamental solution dominates, according to the stationarity and minimum variance criteria, to the α_1-fundamental solution in regions A and C.

Figure 3 illustrates the regions in which the α_1-fundamental and α_2-fundamental solutions are stationary for the case in which $\delta \leqslant 1/2$. In this case, region C is empty because the level curve $\mu_1 = 1$ is outside the relevant area.[6] Therefore, we can distinguish only two regions: region A, where only the α_2-fundamental solution is stationary, and region D, where none of the fundamental solutions is stationary.

The following proposition establishes the conditions under which the backward equilibrium solution (9) is stationary.

PROPOSITION 4. *Let us assume that $\mu_i = \rho_2 + \alpha_i \rho_1$ is a real variable for $i = 1,2$. If $\mu_i < 1$ for $i = 1,2$, then the backward solution, (9), is stationary. In this case, the variance of the endogenous variable characterized by the backward solution, λ_0^b, is given by:*

$$\lambda_0^b = \frac{(1+\mu_1\mu_2)\left(1-\delta^{-1}\right)^2}{(1-\mu_1^2)(1-\mu_2^2)(1-\mu_1\mu_2)}\sigma_v^2 + \frac{\left[(1+\mu_1\mu_2)(1+\rho_2^2) - 2\rho_2(\mu_1+\mu_2)\right]}{(1-\mu_1^2)(1-\mu_2^2)(1-\mu_1\mu_2)}\sigma_u^2. \tag{13}$$

PROOF: See Appendix A.

Figures 2 and 3 can also be interpreted in terms of the stationarity conditions of the backward solution. We show in Appendix B that in the area where the three equilibrium solutions considered exist (all regions but B), the backward solution is only stationary in region C for the case in which $\delta \geqslant 1/2$.[7] Furthermore, for $\delta \leqslant 1/2$, the backward equilibrium solution is not stationary for any combination of ρ_1 and ρ_2 satisfying inequality (10).

According to the definition, stationary and minimum variance criteria, and taking into account the results stated in Propositions 1, 3 and 4, we can classify the combinations of $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$ in the following regions displayed in Figure 2: in region D all three solutions (the two fundamental solutions and the backward solution) exist but none of them is stationary. In region A, only the α_2-fundamental equilibrium solution is stationary. In region C, the three equilibrium solutions are stationary, but the α_1-funda-

6. Notice that for $\delta \leqslant 1/2$ the combination $(\rho_1, \rho_2) = (\frac{1-\delta}{\delta}, \frac{2\delta-1}{\delta})$ implies that $\rho_2 \leqslant 0$.

7. The backward solution is not stationary in region D because $\mu_i > 1$, for $i = 1,2$ and it is also not stationary in region A since $\mu_1 > 1$.

mental solution always has larger variance than the α_2-fundamental solution. Finally, in region B, only the backward solution exists because the fundamental solutions are complex solutions. However, region C is empty for the case in which $\delta \leqslant 1/2$. The following corollaries summarize all these results.

COROLLARY 1. *Let us define $\mu_i = \rho_2 + \alpha_i \rho_1$. Taking into account the definition, stationarity and minimum variance criteria, any combination of $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$ can be classified in one of the following cases according to the selection criteria considered:*

i) *If μ_1 and μ_2 are real and such that $\mu_1 > 1$, and $\mu_2 < 1$, only the α_2-fundamental equilibrium solution is stationary.*

ii) *If μ_1 and μ_2 are complex, the backward equilibrium solution is the only real equilibrium solution.*

iii) *If μ_1 and μ_2 are real and such that $\mu_1 < 1$, and $\mu_2 < 1$, the minimum variance criterion can be used to discriminate between the α_2-fundamental and the backward equilibrium solutions. The result depends on the values of σ_u and σ_v.*

iv) *If μ_1 and μ_2 are real and such that $\mu_1 \geqslant 1$, and $\mu_2 \geqslant 1$, all three equilibria solutions exist, but none is stationary.*

PROOF: This is straightforward from Propositions 1, 3 and 4.

COROLLARY 2. *If $\delta \leqslant 1/2$ there is no combination of $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$, such that the minimum variance criterion can be used to discriminate between the α_2-fundamental and the backward equilibrium solutions, as part iii) in Corollary 1 states.*

PROOF: The proof is straightforward. Notice that $\mu_1 < 1$ implies that $\rho_2 < (2\delta - 1)/\delta$. Therefore, when $\delta \leqslant 1/2$, this inequality implies $\rho_2 < 0$.

Parameters μ_i defined on the above propositions and corollaries can be interpreted by looking at the difference equation system formed by (1) and (2).[8] System (1) and (2) can be expressed in matrix form as:

$$A_0 + A_1 Y_t + A_2 Y_{t+1} + A_3 U_{t+1} = 0,$$

where:

$$Y_t = \begin{bmatrix} y_t \\ z_t \end{bmatrix}, \quad U_{t+1} = \begin{bmatrix} v_{t+1} \\ y_{t+1} - E_t y_{t+1} \\ u_t \end{bmatrix}, \quad A_0 = \begin{bmatrix} -c \\ \rho_0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 1 & -(1-\delta) \\ \rho_1 & \rho_2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\delta & 0 \\ 0 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & \delta & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Operating and rearranging, this matrix system can be written as:

8. We thank an anonymous referee for suggesting this interpretation of the μ_i's.

(14) $$Y_{t+1} = BY_t + CU_{t+1} + D,$$

where $B = -A_2^{-1}A_1,\ C = -A_2^{-1}A_3$ and $D = -A_2^{-1}A_0$.

It is easy to show that μ_i for $i = 1,2$ are the eigenvalues of transition matrix B. In this context the areas illustrated in Figure 2 can be interpreted in terms of the stationarity properties of the difference equation system (14): *(i)* If no equilibrium solution is stationary (that is, in area D), difference equation system (14) is *a source; (ii)* when only the α_2-fundamental solution is stationary (that is, the parameters of ρ_1 and ρ_2 lie on area A), difference equation system (14) is *a saddle path*; *(iii)* if all three equilibrium solutions are stationary (that is, in area C), difference equation system (14) is a *sink;* and *(iv)* along the line $\rho_2 + \alpha_i\rho_1 = 1,\ i = 1,2$; either μ_1 or μ_2 is equal to one, which implies that the endogenous variable and the forcing variables are cointegrated. Moreover, if $\rho_2 + \alpha_i\rho_1 = 1,$ from the second equation of system (5) it is straightforward to show that $\pi_1 = \alpha_i = 1,$ which implies that $\rho_2 + \rho_1 = 1$, and the cointegrated vector is $(1,-1)$. That is, the difference between the endogenous variable and the forcing variable is a stationary process.

Conditional on the five selection criteria considered in this paper, we can illustrate mechanisms that can lead to switches between alternative RE equilibria triggered by small changes in the values of the ρ's. First, consider that condition (10) initially holds and that, according to the minimum variance criterion, the economy is located at a point such as X in Figure 2 where this equilibrium is characterized by the α_2-fundamental equilibrium solution, (7). Now, assume that there is a change in the value(s) of ρ_1 or/and ρ_2 such that condition (10) does not hold (for instance, a switch from X to Y in Figure 2). This change in the value(s) of the parameter(s) characterizing the fundamental process triggers a jump from the equilibrium described by the α_2-fundamental solution, (7), to the equilibrium characterized by the backward solution, (9) (that is, the only RE equilibrium solution that is real). A second mechanism is just the opposite. If condition (10) does not initially hold, then, the initial equilibrium is described by the backward solution, (9). Moreover, according to the minimum variance criterion, a variation in the values of the ρ's implying that (10) is now satisfied, may lead the economy to the equilibrium characterized by the α_2-fundamental solution (7). Finally, a third mechanism leading to switches between the α_2-fundamental and the backward solutions, which does not necessary involve any change in the ρ's, may occur inside region C where all three solutions are stationary. The reason is that, given the values of the ρ's and σ_u^2, the difference between the variances of those solutions (see equations (12) and (13)) depends on the size of σ_v^2, which characterizes the size of the innovations in the fundamental process. This implies that a change in σ_v^2 may lead to switches between two alternative (α_2-fundamental and backward) equilibrium solutions when the equilibrium is selected according to the minimum variance criterion.

All these mechanisms illustrate additional sources of variation in the endogenous variable when there exist multiple RE solutions. In addition to fluctuations in the endogenous variable caused by innovations in the fundamental process, this variable may change due to variations in the parameters characterizing the fundamental process that may lead to switches between alternative RE equilibria.

4 Summary and Implications

In the context of a simple PV model, this paper shows that the presence of a feedback mechanism results in three alternative RE equilibria that we call α_1-fundamental, α_2-fundamental and backward solutions, respectively. Moreover, this paper studies the different dynamic properties displayed by alternative RE equilibria. In particular, we show that the backward equilibrium solution is a '*Lucas* proof' equilibrium since this equilibrium is characterized by the feature of being immune to the *Lucas critique*. Furthermore, we study how some other features displayed by alternative equilibria change for small variations in the parameters characterizing the forcing variable process. An important implication of this analysis is that switches between alternative RE equilibria may arise when there is a variation in the forcing variable parameters because the dynamic features of the alternative equilibria change.

The existence of multiple equilibria itself or combined with the likelihood of switches between RE equilibria introduces several relevant issues for both econometric practice and econometric policy evaluation. For instance, the fact that a small variation in the values of the forcing variable parameters can lead to switches between RE equilibria can be viewed as problematic concerning the issue of expectations coordination. One would argue that a selection criterion based on the stability of a recursive learning algorithm would be more relevant in models with multiple RE equilibria. However, as pointed out by FARMER [2002], it is difficult to model a learning rule that works well for alternative policy regimes. For instance, one can expect estimating a learning rule that works well for a regime characterized by a stable monetary policy, but the same rule is likely to fail in an episode characterized by frequent changes in monetary policy. Alternatively, FARMER [2002] proposes to add a theory of beliefs as a selection device. *Farmer* argues that this strategy may consider a theory of learning to single out an equilibrium, but learning neither explains switches between RE equilibria nor accounts for the dynamic features exhibited by the data, since a theory of beliefs assumes that all learning has already taken place.

The crucial question is, then, *what the theory of beliefs (selection device) is appropriate?* Our own research strategy (VÁZQUEZ [2002] and GUTIÉRREZ and VÁZQUEZ [2000]) essentially consists in letting it be the data determined whose RE equilibria characterize the data generating process in different episodes. Thus, if we do not observe switches between RE equilibria, we need to ask which selection device singles out the RE equilibrium that best fits the data. As pointed out by FARMER [2002], at this point one would like to be able to explain the particular equilibrium as a fixed point of a learning mechanism of the type studied in the learning literature. However, if we observe switches between RE equilibria, we need to ask which of the many possible selection devices is consistent with the pattern of switches observed. In this context, the characterization provided in Section 3 can be useful in sorting among alternative selection devices.

Another issue arises when switches between alternative equilibria are likely to occur. In this case, different equilibria may characterize the dynamics of the endogenous variable at different periods of time. In order to tackle the switching equilibria issue, one may consider identifying the timing of switching-regimes as a first step before analyzing which equilibrium characterizes the evolution of the endogenous variable in a particular subsample (see, for instance, ANDREWS and FAIR [1988] and HAMILTON [1994, Ch. 14] for detailed descriptions of tests for structural change). Another possibility is to consider the change in regime itself as a random variable and empirically characterize the probability law governing the change in regime (see HAMILTON [1994, Ch. 22] for a complete exposition of modeling time series with regime-switches). Using historical US stock market data, the former approach has been followed by GUTIÉRREZ and VÁZQUEZ [2000] to test for the presence of switching equilibria in the context of the PV model of stock prices *with* feedback. The latter approach has been applied by DRIFFILL and SOLA [1998] in the context of the PV model for stock prices *without* feedback.

Another issue arising in a model with multiple RE equilibria is that structural estimation of a particular equilibrium solution does not guarantee an appropriate estimation of the structural parameters unless this equilibrium solution characterizes the true data generating process of the endogenous variable. The reason is that alternative RE equilibria imply different cross equation restrictions. Therefore, the estimation of structural parameters based on a particular set of cross equation restrictions implied by a particular RE equilibrium is meaningful only when those cross equation restrictions are supported by the data. Some well-known rejections of the RE hypothesis can be explained in a context of multiple RE equilibria. For instance, FARMER [2002] explains FAVERO and HENDRY's [1992] rejection of the RE hypothesis as a consequence of multiple RE equilibria.

Another example is the following, CHOW [1989], and BELADI, CHOUDHARY and PARAI [1993] follow the approach of estimating a reduced form equation such as (9) by regressing, y_t on y_{t-1}, z_{t-1} and a constant, then they consider testing the linear restriction characterizing the backward equilibrium solution (see Remark 5) as a way of testing the rational expectations hypothesis. In the light of the analysis carried out in this paper, it is clear that the existence of multiple RE equilibria implies that *(i)* one cannot be sure of identifying structural parameters c and δ of the PV model when a reduced form equation such as (9) is used, and *(ii)* the test of the RE hypothesis suggested by CHOW [1989] is misleading unless one is sure that the endogenous variable is characterized by the backward equilibrium solution (that is, the unique RE equilibrium satisfying that linear restriction). Since this linear restriction characterizes the '*Lucas* proof' backward equilibrium solution, *Chow*'s test should be understood as a test of whether or not the economy is located at an equilibrium which is immune to the *Lucas critique* rather than a test of the RE hypothesis. VÁZQUEZ [2002] uses this type of test in order to study whether historical hyperinflationary episodes are characterized by an RE equilibrium which is immune to the *Lucas critique*. This type of test is relevant from a policy maker point of view because, if agents coordinate their expectations in a '*Lucas* proof' equilibrium (that is, the backward equilibrium solution), then the estimation of a reduced form model can help to evaluate alternative economic polices since the reduced form is policy invariant. The analysis carried out in this paper also illustrates scenarios where the possibility that the economy may switch from a '*Lucas* proof' equilibrium (backward equilibrium solution) to an equilibrium which is not immune to the *Lucas critique* (α_1-fundamental, α_2-fundamental equilibrium solutions) cannot be ruled out. The reason is that some economic policies may involve changes in the parameter values of the forcing variable process that lead to switching equilibria. In short, the *Lucas critique* might be relevant in the near future even though it has not been relevant in the past.

As discussed in Section 3, along the line characterized by $\mu_i = 1$, for $i = 1,2$ the condition $\rho_1 + \rho_2 = 1$ holds and the endogenous and forcing variables are cointegrated. This result combined with the evidence on cointegration provided by a large empirical literature in the context of PV models, led by the seminal paper written by CAMPBELL and SHILLER [1987], suggests the existence of a close relationship between the parameters characterizing the forcing variable process given by $\rho_1 + \rho_2 = 1$. This evidence, thus, suggests that the feedback parameter ρ_1 seems to adjust in a particular manner when the other parameters describing the forcing variable process change. ▼

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APPENDICES

A. *Proofs of Proposition 1, 3 and 4*

Proof of Proposition 1

The proof is straightforward. According to the definition criterion, we consider that an equilibrium solution exists when it is a real rather than a complex solution. When inequality (10) holds all three solutions considered exist. However, when inequality (10) does not hold, α_i is a complex number for $i = 1,2$. Therefore, only the backward solution exists because the two fundamental solutions are complex.

Let us consider the zero-level curve of the term inside the square-root in the definition of α_i:

$$(1 - \delta\rho_2)^2 - 4\rho_1\delta(1-\delta) = 0. \tag{A1}$$

By differentiating this curve, we obtain:

$$\frac{\partial\rho_2}{\partial\rho_1} = -\frac{2(1-\delta)}{1-\delta\rho_2},$$

which is negative for all $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$. Therefore, by the implicit function theorem, we know that there exists a unique continuously differentiable function $\rho_2 = \phi(\rho_1)$, which has negative slope, characterizing the zero-level curve, (A1). It is easy to see that ϕ is a convex function whose intercepts are $(\rho_1,\rho_2) = (0,\delta^{-1})$ and $(\rho_1,\rho_2) = (\frac{1}{4\delta(1-\delta)},0)$. Furthermore, pairs of (ρ_1,ρ_2) in the lower (upper) contour set of (A1) do (not) satisfy inequality (10), (see Figure 1).

For $\delta = 1/5$, (A1) cross the pair $(\rho_1,\rho_2) = (1,1)$. Since $\rho_2 = \phi(\rho_1)$ is a strictly decreasing function, this means that the upper contour set of (A1) does not intersect the region in which $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$. It is easy to prove that for $\delta \leqslant 1/5$, the upper contour set of $\rho_2 = \phi(\rho_1)$ does not intersect the region in which $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$. This means that the region in which the backward solution is the only defined solution is empty for $\delta \leqslant 1/5$.

On the other hand, the combination $(\rho_1,\rho_2) = (1, \frac{1-\delta}{4\delta})$ is on the level curve (A1). Therefore, at least any combination $(\rho_1,\rho_2) = (1-\varepsilon, \frac{1-\delta}{4\delta} - \varepsilon)$ where $0 < \varepsilon < \frac{1-\delta}{4\delta}$ always belongs to the lower contour set of (A1) for any $0 < \delta < 1$. This means that the region in which the three solutions considered are defined is always a non-empty set.

Proof of Proposition 3

Using (2), after recursive substitutions and rearranging, fundamental solutions (6) and (7) can be written as follows:

$$\text{(A2)} \qquad a(L)y_t = k + b(L)v_t + c(L)u_t,$$

where L denotes the lag operator and $a(L) = 1 - \mu_i L$, $b(L) = \alpha_i$, $c(L) = \varphi_i\,(1 - \rho_2 L)$ and $k = \alpha_i \rho_0 + \tau_i\,(1 - \rho_2)$ for $i = 1,2$, respectively.

As is well known, the process characterized by (A2) is stationary if $|\mu_i| < 1$. When μ_i is real, $\mu_i = \rho_2 + \alpha_i \rho_1 \geqslant 0$ for $i = 1,2$, and therefore the condition for fundamental solutions to be stationary is simply $\mu_i < 1$.

Using standard results (for instance, see GRANGER and NEWBOLD [1977, p. 26-27]), the autocovariance generating function for the endogenous variable process (A2), when the process is stationary, can be written as follows:

$$\text{(A3)} \qquad \lambda(L)^i = \frac{b(L)b(L^{-1})}{a(L)a(L^{-1})}\sigma_v^2 + \frac{c(L)c(L^{-1})}{a(L)a(L^{-1})}\sigma_u^2.$$

The variance of the endogenous variable processes characterized by fundamental solutions (6) and (7) (λ_0^i, for $i = 1,2;$ respectively) is equal to the coefficient associated with L^0 in the power series expansion of the autocovariance generating function $\lambda(L)^i$, which, after some algebra, can be written as:

$$\text{(A3)} \qquad \lambda_0^i = \frac{\alpha_i^2}{1-\mu_i^2}\sigma_v^2 + \frac{1+\rho_2^2 - 2\rho_2\mu_i}{1-\mu_i^2}{\varphi_i}^2\sigma_u^2.$$

Notice that $\mu_i = \rho_2 + \alpha_i \rho_1$ and $\varphi_i = 1/(1 - \alpha_i \rho_1 \delta)$. Therefore, in order to show that the variance of α_2-fundamental solution (7) is lower than the variance of α_1-fundamental solution (6), it is sufficient to show that λ_0^i is increasing in α_i, since $\alpha_1 \geqslant \alpha_2$. Let us denote the first and second terms in (A3) by A and B, respectively. Then:

$$\frac{\partial \lambda_0^i}{\partial \alpha_i} = \frac{\partial A}{\partial \alpha_i} + \frac{\partial B}{\partial \alpha_i}.$$

Operating we can obtain that:

$$\frac{\partial A}{\partial \alpha_i} = \frac{2\sigma_v^2 \alpha_i\,(1-\mu_i\rho_2)}{\left(1-\mu_i^2\right)^2} > 0,$$

$$\frac{\partial B}{\partial \alpha_i} = \frac{2\rho_1\varphi_i^2\sigma_u^2}{\left(1-\mu_i^2\right)}\left\{\frac{\delta\left(1-\mu_i^2\right)+\alpha_i\rho_1}{(1-\delta\alpha_i\rho_1)} + \frac{\mu_i\alpha_1^2\rho_i^2}{\left(1-\mu_i^2\right)}\right\} > 0.$$

Notice that those partial derivatives are positive because all terms in parenthesis are positive under the stationarity condition. This implies that the variance of the endogenous variable is increasing in α_i. This completes the proof.

Proof of Proposition 4

The backward equilibrium solution, (9), with $\varepsilon_t = 0$ can be written as an ARMA process as follows. By taking into account the forcing variable process, equation (2), the backward solution can be written as:

$$y_t = \left(1-\delta^{-1}\right)\rho_0 + \delta^{-1} y_{t-1} + \left(1-\delta^{-1}\right)\rho_1 y_{t-2} + \left(1-\delta^{-1}\right)\rho_2 z_{t-2} + \left(1-\delta^{-1}\right) v_{t-1} + u_{t-1}.$$

Adding and subtracting $\rho_2 y_{t-1}$ from this and using (9), we have that:

$$y_t = \left(1-\delta^{-1}\right)\rho_0 + \left(\delta^{-1}+\rho_2\right) y_{t-1} + \left[\rho_1 - \delta^{-1}\left(\rho_1+\rho_2\right)\right] y_{t-2} + \left(1-\delta^{-1}\right) v_{t-1} - u_{t-1} - \rho_2 u_{t-1},$$

or alternatively:

$$q(L) y_t = \left(1-\delta^{-1}\right)\rho_0 + m(L) v_t + n(L) u_t, \tag{A4}$$

where $q(L) = 1 - (\delta^{-1}+\rho_2)L - \left[\rho_1 - \delta^{-1}\left(\rho_1+\rho_2\right)\right] L^2$, $m(L) = \left(1-\delta^{-1}\right) L$ and $n(L) = (1-\rho_2 L) L$.

Since the backward solution, (9), can be expressed as the ARMA (2,1) process (A4), this equilibrium solution is stationary whenever all roots of $q(L) = 0$ lie outside the unit circle. Let us denote by q_1 and q_2 the roots of $q(L) = 0$. Thus,

$$q(L) = \left[\delta^{-1}\left(\rho_1+\rho_2\right) - \rho_1\right](L-q_1)(L-q_2) = 0,$$

where:

$$q_1 = \frac{(1+\rho_2\delta) + \sqrt{(1-\rho_2\delta)^2 - 4\rho_1\delta\left(1-\delta\right)}}{2\left(\rho_1+\rho_2-\delta\rho_1\right)},$$

$$q_2 = \frac{(1+\rho_2\delta) - \sqrt{(1-\rho_2\delta)^2 - 4\rho_1\delta\left(1-\delta\right)}}{2\left(\rho_1+\rho_2-\delta\rho_1\right)}.$$

Notice that the square root in the definition of the q's is the same as the one in the definition of the α's. Therefore, we can establish the following relationship between these q's and the α's that define the fundamental solutions:

$$q_i = \frac{1}{\rho_2 + \alpha_i \rho_1} \equiv \frac{1}{\mu_i},$$

for $i = 1,2$. Since $\alpha_i \geqslant 0$ (for $i = 1,2$) when inequality (10) holds, the condition for the backward solution to be stationary is that $\mu_i < 1$, for $i = 1,2$.

In order to obtain the variance of the backward solution when the process is stationary, let us rewrite $q(L)$ in (A4) as follows:

$$q(L) = (1 - \mu_1 L)(1 - \mu_2 L),$$

where $0 < \mu_i = q_i^{-1} < 1$. The autocovariance generating function for the endogenous variable process under the backward solution, $\lambda(L)^b$, can be written as follow:

$$\lambda(L)^b = \sigma_v^2 \frac{m(L)\, m\left(L^{-1}\right)}{q(L)\, q\left(L^{-1}\right)} + \sigma_u^2 \frac{n(L)\, n\left(L^{-1}\right)}{q(L)\, q\left(L^{-1}\right)}$$
$$= \frac{\sigma_v^2 \left(1 - \delta^{-1}\right)^2 + \sigma_u^2 (1 - \rho_2 L)\left(1 - \rho_2 L^{-1}\right)}{(1 - \mu_1 L)(1 - \mu_2 L)(1 - \mu_1 L^{-1})(1 - \mu_2 L^{-1})}.$$

The variance of the endogenous process characterized by the backward solution (A4), is derived from the coefficient associated with L^0 in the power series expansion of the autocovariance generating function $\lambda(L)^b$. After simple, but tedious, algebra, one can show that the variance of stock prices process is given by equation (13). This completes the proof.

B. *Analysis of Figures 2 and 3 (Section 3)*

Let us consider the level curve:

(B1) $$\mu_1 \equiv \rho_2 + \alpha_1 \rho_1 = 1.$$

By differentiating this curve, we obtain:

(B2) $$\frac{\partial \rho_2}{\partial \rho_1} = \frac{2(1 - \delta)}{-(1 - \rho_2 \delta) + \sqrt{(1 - \rho_2 \delta)^2 - 4\delta (1 - \delta)\, \rho_1}}.$$

Taking into account that along (B1),

(B3) $$2\delta - (1 + \delta \rho_2) = \sqrt{[1 - \delta(\rho_1 + \rho_2)]^2 - 4\rho_1 \rho_2},$$

we can see that (B2) is:

$$\frac{\partial \rho_2}{\partial \rho_1} = -1.$$

Therefore, the implicit function associated with (B1) is a linear function with negative slope. On the other hand, $(\rho_1,\rho_2) = (\frac{1-\delta}{\delta}, \frac{2\delta-1}{\delta})$ lies on (B1). Notice that this pair also satisfies the zero-level curve (A1) defined in Proposition 1:

$$(1-\delta\rho_2)^2 - 4\rho_1\delta\,(1-\delta) = 0 \Longleftrightarrow \mu_1 = \mu_2.$$

Moreover, since the term on the right-hand side of (B3) is positive, all pairs of (ρ_1,ρ_2) in the level curve (B1) must satisfy:

$$\text{(B4)} \qquad \rho_2 \leqslant \frac{2\delta-1}{\delta}.$$

Notice that the pair $(\rho_1,\rho_2) = (\frac{1-\delta}{\delta}, \frac{2\delta-1}{\delta})$ satisfies (B4) with equality.

All these results mean that the level curve (B1) is equivalent to the set of all pairs (ρ_1,ρ_2) on the line $\rho_1+\rho_2 = 1,$ such that $\rho_2 \leqslant \frac{2\delta-1}{\delta}$. In other words, in Figures 2 and 3 the level curve $\mu_1 = 1$ is equivalent to the segment connecting the points $(\rho_1,\rho_2) = (\frac{1-\delta}{\delta}, \frac{2\delta-1}{\delta})$ and $(\rho_1,\rho_2) = (1,0)$.

Furthermore, it is easy to see that:

$$\text{(B5)} \qquad \frac{\partial\rho_2+\alpha_1\rho_1}{\partial\rho_1} < 0.$$

This means that all pairs satisfying inequality (10) and located to the right (left) of the level curve (B1) satisfy $\mu_1 < 1$ ($\mu_1 > 1$).

On the other hand, for any combination satisfying inequality (10),

$$\mu_1 = \rho_2+\alpha_1\rho_1 \geqslant \frac{1+\delta\rho_2}{2\delta}.$$

Therefore, all pairs satisfying inequality (10) and located northwest of combination $\left(\frac{1-\delta}{\delta}, \frac{2\delta-1}{\delta}\right)$ are such that:

$$\rho_2 > \frac{2\delta-1}{\delta} \quad \Longrightarrow \frac{1+\delta\rho_2}{2\delta} > 1 \quad \Longrightarrow \mu_1 = \rho_2+\alpha_1\rho_1 > 1.$$

This result and (B5) imply that, only for those pairs satisfying inequality (10) and located to the right of the level curve (B1) the α_1-fundamental solution is stationary.

Following the same steps, we can see that the level curve $\mu_2 \equiv \rho_2+\alpha_2\rho_1 = 1$ is equivalent to the set of all pairs (ρ_1,ρ_2) on the line $\rho_1+\rho_2 = 1,$ such that $\rho_2 \geqslant \frac{2\delta-1}{\delta}$. In other words, in Figures 2 and 3 the level curve $\mu_2 = 1$ is equivalent to the segment connecting the points $(\rho_1,\rho_2) = (\frac{1-\delta}{\delta}, \frac{2\delta-1}{\delta})$ and $(\rho_1,\rho_2) = (0,1)$.

Moreover, it is easy to prove that the α_2-fundamental solution is not stationary only for those pairs satisfying inequality (10) and located on and above that level curve.

C. *A Simple Growth Model*

Consider the simple optimal growth model:

$$\max E_t \sum_{t=0}^{\infty} \beta^t \log C_t$$

$$C_t + K_{t+1} = Z_t K_t^{\theta} + (1-\kappa) K_t,$$

$$\log Z_t = (1-\psi)\log \overline{Z} + \psi \log Z_{t-1} + \varepsilon_t,$$

with K_0 and Z_0 given. C_t is consumption, K_t is capital and Z_t is total factor productivity; θ and κ are parameters belonging to the interval (0,1) denoting capital share and depreciation rate, respectively; $0 < \psi < 1$ and $\overline{Z}$ are also parameters and ε_t is a white noise process.

It is well-known that there are no explicit closed-form solutions for K_{t+1} and C_t for this model unless $\kappa = 1$ (that is, there is complete depreciation of capital). Solving the model then requires the use of a numerical approximation method. Since the utility function is assumed to be logarithmic a sensible approach to solve the model is to consider a log-linear approximation. Consider, for instance, UHLIG's [1999] log-linear method. This method amounts to firstly log-linearizing the necessary equations characterizing the equilibrium to make the equations approximately linear in the log-deviations from the steady-state, and secondly solving the recursive equilibrium law of motion governing the endogenous variables of the model through the method of undetermined coefficients. Let us denote the steady-state value of a variable X_t by $\overline{X}$ and the log-deviation from its steady-state value by the lower-case letter x_t. Formally, $x_t = \log X_t - \log \overline{X}$. The log-linearized first-order conditions characterizing the equilibrium are:

$$c_t = \frac{\overline{Y}}{\overline{C}} z_t + \frac{\overline{K}}{\beta \overline{C}} k_t - \frac{\overline{K}}{\overline{C}} k_{t+1},$$

$$r_t = [1-\beta(1-\kappa)][z_t - (1-\theta)k_t],$$

$$E_t(c_t - c_{t+1} + r_{t+1}) = 0,$$

$$z_t = \psi z_{t-1} + \varepsilon_t,$$

where $r_t = \log R_t - \log \overline{R}$, and R_t is *the capital return* defined by $R_t = \theta Z_t K_t^{\theta-1} + (1-\kappa)$. After some tedious, but simple, algebra one can show that the recursive law of motion for the simple optimal growth model is given by the following set of equations:

$$k_{t+1} = \gamma_{kk} k_t + \gamma_{kz} z_t,$$

$$r_t = \gamma_{rk} k_t + \gamma_{rz} z_t,$$

$$c_t = \gamma_{ck} k_t + \gamma_{cz} z_t,$$

where the γ_{ij} coefficients are related to the structural parameters as follows:

$$\gamma_{kk} = \frac{\gamma}{2} - \sqrt{\left(\frac{\gamma}{2}\right)^2 - \frac{1}{\beta}},$$

$$\gamma_{rk} = -(1-\theta)[1-\beta(1-\kappa)],$$

$$\gamma_{ck} = (\frac{1}{\beta} - \gamma_{kk})(\frac{\overline{K}}{\overline{C}}),$$

$$\gamma_{rz} = 1 - \beta(1-\kappa),$$

$$\gamma_{kz} = \frac{\gamma_{rz}\psi + (1-\psi)\frac{\overline{Y}}{\overline{C}}}{\gamma_{ck} - \gamma_{rk} + (1-\psi)(\frac{\overline{K}}{\overline{C}})},$$

$$\gamma_{cz} = \frac{\overline{Y}}{\overline{C}} - (\frac{\overline{K}}{\overline{C}})\gamma_{kz},$$

where $\gamma = \frac{(1-\theta)[1-\beta(1-\kappa)][1-\beta+\beta\kappa(1-\theta)]}{\theta\beta} + 1 + \frac{1}{\beta}$, $\overline{R} = \beta^{-1}$, $\overline{K} = \left(\frac{\theta\overline{Z}}{\overline{R}-1+\kappa}\right)^{\psi-1}$, $\overline{Y} = \overline{ZK}^{\theta}$, and $\overline{C} = \overline{Y} - \kappa\overline{K}$. The expressions of the γ_{ij} coefficients show that changes in ψ (that is, the parameter characterizing the forcing variable process Z_t) change some of the reduced form parameters of k_{t+1} and c_t, but do not affect the reduced form parameters of r_t. Moreover, with complete depreciation (that is, $\kappa = 1$), one can easily show that $\gamma_{kk} = \gamma_{ck} = \theta$, $\gamma_{rk} = -(1-\theta)$, and $\gamma_{kz} = \gamma_{rz} = \gamma_{cz} = 1$, which implies that the reduced form parameters in the case of complete depreciation of capital are invariant when the forcing variable parameter ψ changes. In particular, we have that $c_t = \theta k_t + z_t$, which implies that $C_t = \frac{\overline{C}}{\overline{Y}} Y_t = (1-\theta\beta)Y_t$. In words, the consumption function is invariant to changes in the parameters of the total factor productivity process whenever there is complete depreciation. This result implies that the consumption function is immune to the *Lucas critique* in this special case.