# Multiple Equilibria and Identification » by Russel Cooper: A Comment

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# **1** Introduction

In his lecture at the Conférence des Annales, Russel W. COOPER raises issues that certainly will become more and more important as models that exhibit multiple equilibria become more and more common in today economic literature. These models are appealing as they may allow economists to cast some light for instance on the mechanisms that govern the large swings in growth pace or on the reasons why similar institutions can lead to different economic situations. Russel made decisive contributions to understand some of these mechanisms. Nevertheless, a concern exists that due to their multiple equilibria, these models might not be falsified. Russel W. COOPER presents various empirical examples to illustrate that during the last ten years progress has been made. It seems possible to draw some empirical knowledge from these estimation exercises. Notwithstanding, complete empirical investigations of these models are rare because their estimating remains difficult and economists have to resort to simplifying assumptions or calibration approach to avoid these difficulties. In this comment, at the risk of distortion, I will try to restate some of Russel W. COOPER's points focusing on their statistical dimensions and emphasizing that recent progress in econometrics may help to tackle some of these issues.

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# 2 Multiple Equilibria and Statistical Identification

Multiple equilibria in economics and identification issues in statistics are in a certain sense dual. For a completely specified model characterized by a vector of parameters  $\theta$ , the existence of multiple equilibria corresponds to the fact that for a given value of  $\theta$  and a given chronology of predetermined variables  $(x_t)_t$  and shocks  $(\varepsilon_t)_t$  drawn in a given distribution  $Q_{\theta}(\varepsilon)$ , several equilibrium values of the endogenous variables  $(y_t)_t$  are possible. In other terms, the function of  $\theta$ ,  $x_t$  and  $\varepsilon_t$  that defines the equilibrium values is a correspondence, (we denote it in the sequel  $y_t \in C(x_t, \varepsilon_t, \theta)$ .) In statistics, a parametric model is not identified when different values of  $\theta$  do not define different probability distributions of the endogenous variables conditionally on the exogenous ones  $P_{\theta}(y|x)$ . The statistician notices that when working on data generated by a model with multiple equilibria, he cannot in general without additional information define a probability of observing a particular value in the correspondence C. He must deal with a semi-parametric model in the sense that he must consider all the distributions F(y|x) that are compatible with the set of conditions : there exists an admissible  $\theta$ , such that  $\forall t \in \{1, \dots, T\}, y_t \in C (x_t, \varepsilon_t, \theta)$  where the distribution of  $\varepsilon_t$  is  $Q_{\theta}$ .

This duality "several equilibrium values for one parameter value" and "several parameter values consistent (in a certain sense) with the observed endogenous variables" is formal. It does not tell us *a priori* if statistical identification is more difficult in case of multiple equilibria. JOVANOVIC [1989] in a relatively general framework shows that non-uniqueness of the equilibrium because it makes larger the set of F(y|x) consistent with the structure of the model and the distribution of  $(\varepsilon_t)_t$ , makes lack of identification easier.

### **3** A Semi-Parametric Approach

Basically, the statistician is working with a semi-parametric framework. In these conditions, (s)he can use a GMM-type approach. In this set-up, the statistical model is (weakly) identified when the set of identifying equations has a unique solution. The question of importance is then the choice of these equations. Nevertheless, approaches such as that introduced by GALLANT and TAUCHEN [1996] cannot be used because the structural model does not define a data generating process. Without additional information, GMM can be used in particular cases such as models in which the equilibrium values are given by an equation that admits several solutions but in which the exogenous shocks can be univocally isolated. If we have the following relation:

$$y_t \in C (x_t, \varepsilon_t, \theta) \iff H (y_t, x_t, \theta) = \varepsilon_t$$

 $\theta$  can be estimated with the set of the empirical counterparts of moments such as  $E_{\theta,x}x'_t H(y_t,x_t,\theta) = 0$ . This means in particular that the way the equilibrium value is in practice selected does not affect these moments. Identification amounts then to the property that a unique value of  $\theta$  satisfies the last equation, this depends heavily on the model specification and it seems difficult to comment on without additional assumptions. However, let us remark that this set of identifying equations relies on the structural assumption about the exogeneity of  $\varepsilon_t$ . From an empirical point of view, this assumption may be imperfect and part of the noise ( $\varepsilon_t$ ) may be related to this default. According to the degree of difficulty we can expect in the coordination process, the empirical relevance of this assumption might be reassessed.

### 4 A Parametric Approach

Otherwise, approaches based on probabilistic distribution of the values taken by  $y_t$  cannot be used without the specification of a set of selection devices (possibly a singleton). The specification of these selection devices can be set in different manners according to modeler's objectives when working with aggregate data but seems to be overlooked when working with individual data. Russel W. COOPER presents various examples I propose to comment on from a statistical point of view and according to the use of the model.

The selection devices can be deterministic or stochastic. In both cases, once their specification has been chosen, the statistical model is as usual. The basic questions then become: *are these selection devices of interest for the modeler? How can they be specified in practice?* As introduced by Russel W. COOPER, we can expect that parameters associated to the selection device have to be introduced. The above questions can then be rephrased as follows considering the status of these additional parameters: *are they nuisance parameters or parameters of interest?* This issue is of course related to the question analyzed by the modeler.

#### 4.1 The Equilibrium Selection Device is of Interest

If the model has to be used in forecasting exercise or economic policy analysis, the way the equilibrium is selected is a key input, *additional para*-*meters*  $\theta^+$  *are parameters of interest*. Once the selection device specified, the modeler is considering a classical completely specified structural model. Identification of  $\theta$  may result from the chosen selection device, but this is not an issue as all the elements of the model are structural.

Nevertheless, designing the selection device is a complicated issue. When working with only macroeconomic time series, it seems difficult to describe the way people coordinate on one possible outcome without referring to their beliefs, their information set and their relative heterogeneity, except may-be when a sunspot selection mechanism with its sunspot variable is postulated. FARMER and GUO [1995] follow implicitly this road with a sunspot variable that is not observed by the econometrician. Interpretation of the result in this framework remains fragile.

#### 4.2 The Equilibrium Selection Device is not of Interest

The additional parameters may be considered as nuisance parameters when for instance the modeler is interested in some particular parameters of the initial model that correspond to a theoretical issue or wants to analyze the consequences of various policy changes in the neighborhood of each equilibrium. JOVANOVIC [1989] shows that the restriction to a large set of extrinsic random selection device in comparison with all the possible selection devices is in general<sup>1</sup> of no consequence on the identifiability properties of the initial model. Under the additional assumption that the selection device is stationary conditionally on the set of available observations at each date, it can then be descriptively specified without referring to a particular coordination scheme. In the simplest cases, it may involve a latent variable that captures the result from the coordination scheme in selecting one particular equilibrium. When we expect some persistence in the equilibrium selection, Hidden Markov Chain model is a natural candidate<sup>2</sup>. This is the approach followed by COOPER and CORBAE [2001]. A latent discrete variable describes the change in agents' expectations that are labelled by 'optimism' and 'pessimism'.

If the model was initially identified, this additional mechanism allows the modeler to estimate the parameter of interest. Nevertheless, this particular choice may be non-neutral on the quality of the estimates. The class of HMM does not capture all the distributions consistent with the initial structure and selecting *a priori* a particular device may lead to estimates of poor quality with large asymptotic variance. Without any particular knowledge on the device, a possible estimation strategy may correspond to the use of a Bayesian framework with diffuse priors on a large class of extrinsic random selection devices.<sup>3</sup> Technically, this may be complicated. When a true stationary selection device exists and can be correctly approximated by one element in the class under study, a posteriori distributions give an indication on the most relevant selection device. Nevertheless, with the usual sample size in macroeconometric studies, by averaging on a large set of possible mechanisms, we may end up with not so accurate estimates. This makes the rejection of the model more difficult and the ability to compare models with different numbers of equilibria weak.

On the other hand, *identifiability may result from the choice of a particular selection device model*. This naturally introduces a source of extrinsic noise whose chosen specification affect value of and inference on the parameters of interest.

Mechanisms introduced by COOPER and CORBAE [2001] to explain American Great Depression are clever and credible. Technically, the model is

<sup>1.</sup> In case of atomless distribution of  $(\varepsilon_t)_t$ 

<sup>2.</sup> If the number of equilibria is fixed.

<sup>3.</sup> We can restrict our attention to those that satisfy a priori some meaningful properties.

singular from a probabilistic point of view in the sense that the exogenous shock dimension is smaller than the endogenous variable one. It may be in practice difficult without additional assumptions to establish that the model is identified for any selection device.

In the approach considered in this section, the modeler mixes a structural model (the initial model) and a descriptive one (the selection device). This strategy may be the more relevant considering the difficulty designing a coordination mechanism on aggregate data but it may be costly for the inference properties of the estimates and require worked-out estimation procedures.

# 4.3 Models with non-unique equilibrium are difficult to estimate

For any stochastic selection device, the computation of the estimates is in general difficult as the number of equilibrium may depend on the current and past values of  $(x_t, \varepsilon_t)$  and on the nature of the equilibrium selected in the recent past. This may necessitate the use of simulation methods as indicated by Russel W. COOPER. According to the degree of complexity, some may wonder if the cost of estimating a complete model under a lot of assumptions may be justified as soon as we accept the idea that any macroeconomic model is always a simple (simplistic !) sketch of complex economic phenomena that is useful to answer a finite set of questions. Calibration is a way to avoid this difficulty and may allow the modeler to focus on particular features that are characteristics of the model under study. Nevertheless, it does not allow him to statistically judge of the fit.<sup>4</sup>

In this respect, the results obtained by COOPER and CORBAE [2001] are illustrative and show that reasonable parameter values are consistent with some key empirical measures, but it can hardly be said more<sup>5</sup>. We cannot deduce a measure of the uncertainty that exists on each of these parameter values. On the other hand, the value of the HMM endogenous variable that selects the equilibrium in their paper is assumed to be known for various dates as its interpretation in terms of 'optimism' and 'pessimism' is natural and in keeping with the common sense. This allows for the calibration exercise. Without such assumptions, simulations of the model to derive the moments on which is based the calibration would be necessary. The transition probability values of the Markov chain matrix would then be of importance and might make the conclusion less convincing.

A measure of fit for calibrated singular models has been proposed by WATSON [1993]. It consists in augmenting the dimension of shocks so that the model can match the second order moments of the actual data. The size of these errors is used to derived a measure of fit. When the model is not singular, a possible alternative method consists in using indirect inference

<sup>4.</sup> We should keep in mind that even in completely specified statistical models, goodness-of-fit tests are not the panacea. BICKEL, RITOV and STOKER [2001] show that omnibus goodness-of-fit tests have little power and will not reject until the lack of fit is extreme.

<sup>5.</sup> This also is the case in MORO [2000] on which Russel W. COOPER also comments. In this paper, identification results from assumptions on the functional link between the different elements in the model. Calibration is done under the assumption that only one equilibrium is observed. It seems difficult to assess the fit of the model in this condition.

method as introduced by SMITH [1993] and GOURIÉROUX, MONFORT and RENAULT [1993] to learn from good-fitting descriptive models some parameters of interest in the (possibly ill-fitting) structural model. It may be in particular a way of introducing a statistical dimension in the calibration approach (DRIDI and RENAULT [1999]).

## 5 Microeconomics

Russel W. COOPER restricts his attention to interaction models with strategic complementarities. In general, these empirical models aim at analyzing agents' behavior (and possibly assessing the consequences of a change in policy). Once the structural parameters estimated, the modeler can study if the model he considers has multiple equilibria. Individual equation can be seen as a best response to an aggregate value of the strategic variable. A first approach corresponds to solving for the equilibrium value of the aggregate variables and to substituting them in the individual equation. This is not always possible in case of non-linear models. In a second approach, the econometrician works on the individual equations. The observed aggregate value is interpreted as the equilibrium one and considered as exogenous.<sup>6</sup> Standard estimation methods can then be used. When this exogeneity assumption is less relevant, for instance when the aggregate value corresponds to measures on small neighborhoods or is measured with errors, instrumental variables have to be used.

In these models, lack of identification is related to the difficulty disentangling the sources of close behaviors among people belonging to the same neighborhood. These interactions can be endogenous or contextual (in MANSKI's words [1993]). The lack of identification is present in general in linear static specification of the best response equation (MANSKI [1993]). As soon as we consider non-linear functional forms, endogenous selection in the neighborhood or dynamic specification, the lack of identification situation is most of the time a zero-measure set (BROCK and DURLAUF [2000]). Working with individual data seems more comfortable as long as interest is limited to the structural parameters.

# 6 Conclusion

As we can see, estimating as well as checking the identifiability conditions in completely specified structural models with non-unique equilibrium is a

<sup>6.</sup> This corresponds to the idea that in a static specification the equilibrium value that results from the individual actions is not affected on average by the idiosyncratic error terms thank to the law of large numbers. It is valid if we consider that all the agents coordinate on a particular equilibrium.

complicated task. The macroeconomic examples considered by Russel W. COOPER do not adopt a complete statistical framework and do not consider the identification issue. This may weaken the conclusion they reached. Recent developments in econometrics seem to be useful to construct a more controllable framework, allowing the modeler to focus on the key inputs in the model without losing the possibility of judging of its performance. On the other hand, working with individual data may simplify some issues as long as interest is limited to the structural parameters.

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