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Increasing Returns to Scale and Nonlinear Endogenous Fluctuations in a Simple Overlapping Generations Model: A Pedagogical Note

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ABSTRACT. – We study the role of increasing returns (in production) on the emergence of nonlinear endogenous fluctuations in a simple overlapping generations model. We specify conditions for the occurrence of local flip and Hopf bifurcations in terms of relevant parameters. We, also, present orbits exhibiting endogenous fluctuations, for some numerical examples. Our results, in compliance with other recent works, suggest that the environment in which endogenous fluctuations can emerge approaches those considered more plausible when there are increasing returns to scale.

Rendements croissants d'échelle et fluctuations endogènes non linéaires dans un modèle simple à générations imbriquées : une note pédagogique

RÉSUMÉ. – Nous étudions les effets des rendements croissants (dans la production) sur l'émergence de fluctuations endogènes non linéaires, dans un modèle simple à générations imbriquées. Les conditions garantissant l'occurrence des bifurcations locales flip et Hopf sont données sur un ensemble de paramètres pertinents. Nous présentons quelques exemples numériques d'orbites avec fluctuations endogènes. Les résultats, en accord avec d'autres études récentes, suggèrent que l'environnement économique favorable à l'émergence des fluctuations endogènes s'approche de ceux considérés comme plausibles en situation de rendements croissants d'échelle.

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1 Introduction

The current literature in macroeconomics usually considers general equilibrium dynamic models for economies with infinite horizon. Typically, in these models explicit dynamics are obtained by considering that agents solve intertemporal optimization problems. Rational expectations and market clearing are also commonly used assumptions. There are, however, important differences between the models explored. The most relevant one may the emphasis in different sources of persistent and bounded fluctuations in macro variables.

A great set of works considers that economic fluctuations are generated by *exogenous* shocks to the economic system. These works usually assume a traditional economic structure, where equilibrium trajectories converge to a steady state solution along which the variables' values remain constant, or else the trajectories would become explosive. Obviously, in this context, stationary persistent fluctuations would not be observed. But the existence of repeated stochastic exogenous shocks to the fundamentals of the model (technologies, preferences or economic policy variables), by persistently changing the steady state equilibrium point where otherwise the economy would rest, may be a source of fluctuations. Nevertheless, an increasing body of recent works has been showing that economic fluctuations may be *endogenous* to the dynamic system, emerging even in the absence of exogenous shocks to the fundamentals of the model. These works show that bounded fluctuations around a steady state solution, driven by self-fulfilling volatile expectations, are compatible with intertemporal optimizing behavior and market clearing in all periods, even in an environment without fundamental or intrinsic uncertainty, as soon as we depart from "*classical*" assumptions.[1]

Endogenous fluctuations may have a stochastic nature (sunspots) if some (extrinsic) uncertainty in expectations does exist. Stochastic endogenous fluctuations may even emerge in linear models. However, deterministic endogenous fluctuations, along which persistent and bounded fluctuations are observed in the absence of any kind of uncertainty, only emerge in the presence of nonlinearities exhibited by the dynamic system. Local bifurcation analysis is an appropriate tool to study the emergence of these fluctuations around a steady state solution. In rough terms, a local bifurcation occurs when, by slightly varying some parameter of the system, the local stability properties of the steady state are changed. In nonlinear systems, the occurrence of flip and Hopf bifurcations generates the existence of equilibrium

1. Several different assumptions engender the necessary mechanisms for the existence of equilibrium trajectories exhibiting bounded expectations-driven fluctuations: preferences with strong income effects or heavy future discounting, technologies exhibiting low substitution between labor and capital, increasing returns or differences across sectors, or the existence of market imperfections. See, for instance, among others, J.-M. GRANDMONT [1985], P. REICHLIN [1986], P. REICHLIN [1992], C. D'ASPREMONT, R. DOS SANTOS FERREIRA and L.-A. GÉRARD-VARET [1995], J. GALI [1994], R. FARMER and J.-T. GUO [1994] , J. BENHABIB and R. FARMER [1994], and A. VENDITTI [1998]. The reader may also benefit from reading the surveys of M. BOLDRIN and M. WOODFORD [1990], R. GUESNERIE and M. WOODFORD [1992] and A. VENDITTI [1996].

trajectories exhibiting endogenous fluctuations. The main idea is that nonlinearities become able to bound trajectories that would become explosive if the system were linear.

Our aim is to analyze the influence of increasing returns to scale on the emergence of nonlinear endogenous fluctuations, through the occurrence of local bifurcations, by considering a simple overlapping generations model.[2]

In the OLG model considered there is a unique asset, either money or productive capital, and households work only when young and consume when old. We assume increasing returns to scale at the firm level, by specifying a Leontief technology homogeneous of degree $\rho > 1$. In our model, there is imperfect competition with free entry and exit in the output market. However, the market power vanishes when $\rho = 1$, as the zero profits condition leads in this case to an infinite number of firms. Thus, the assumption of perfect competition is naturally recovered as the limit case of constant returns to scale.[3] Our model can be seen as a generalization of the models used by J.-M. GRANDMONT [1985], P. REICHLIN [1986], R. DE VILDER [1996], or by M. WOODFORD [1986].[4] In these works the standard assumptions of perfect competition and constant returns to scale in production are considered.

In our paper, we establish conditions on relevant "*parameters*", such as the elasticity of labor supply, the capital share of output and the degree of returns to scale, so that flip and Hopf bifurcations generically occur. We, then, check whether or not these conditions are satisfied for empirically plausible values of the referred parameters. We also obtain, through computer simulations, numerical examples of these bifurcations and of trajectories exhibiting endo-

2. In this paper, we are mainly concerned with the emergence of deterministic endogenous fluctuations through the occurrence of local bifurcations in nonlinear systems. However, nonlinearities and bifurcations are also important for the emergence of stochastic endogenous fluctuations. See, J.-M. GRANDMONT, P. PINTUS and R. DE VILDER [1998]. This explains why we use the term "*nonlinear*" instead of deterministic endogenous fluctuations.

3. Indeed, the assumption of perfect competition is not compatible with increasing returns to scale at the firm level, as marginal cost pricing leads in this case to negative profits. However, we could have assumed perfect competition if increasing returns were due to external effects on firms. The reader may latter check that our results would equally apply if we had considered perfect competition and constant returns to scale at the firm level and additionally had assumed that inputs originate an external effect proportional to their respective share in total income. In spite of that, the assumption of imperfect competition in dynamic macroeconomic models has recently been quite used, either to understand the effects of demand policy, the emergence of unemployment, its implications for the dynamics of capital accumulation or even to explain the features of business cycles. See, the survey of J. SILVESTRE [1995]. The joint assumptions of imperfect competition and increasing returns have also been widely explored since R. HALL [1988] has shown evidence for its empirical support. Moreover, the assumption of free entry and exit in the output market allows the consideration of a zero profit equilibrium, which seems appealing, given empirical evidence favoring the absence of significant pure profits, at least for the US economy. See, J. ROTEMBERG and M. WOODFORD [1993].

4. WOODFORD considers constant returns to scale in a model where agents have an infinite horizon for lifetime, instead of an OLG model. However, in his model the dynamics become very simple. Due to financial markets' imperfections, current wages (or real money holdings) are spent in consumption and capital rentals are accumulated in capital goods. Indeed, when the technology is of a Leontief type, it turns out that the dynamics thus generated are identical to those in our OLG economy if we assumed constant returns to scale. The main difference of our model is the consideration of increasing returns to scale. Note, also, that the period's length in our OLG model can then be made compatible with business cycles frequencies, since our OLG structure generates identical dynamics to a model where agents have an infinite horizon (the Woodford model).

genous fluctuations. These experiments are also a simple way to detect whether these trajectories are attracting nearby points or repelling them.

The results suggest, as expected, that increasing returns may facilitate the emergence of nonlinear endogenous fluctuations, once the existence of increasing returns to scale may enhance the plausibility of the environment in which endogenous fluctuations can emerge. As the degree of returns to scale increases, the range of parameters' values for which flip and Hopf bifurcations occur either moves or shrinks, respectively, in the direction of the range usually considered to be empirically plausible. Estimates for the capital share of output point to values between 0.25 and 0.4 in industrializes countries, and empirical estimations for the labor supply elasticity point to values around zero (biased to positive values).[5] As we shall see, the elasticity of labor supply must take values close to –0.5 in order that a flip bifurcation occur with constant returns to scale, while under increasing returns to scale they occur for less negative elasticities of labor supply. Moreover, we construct a flip bifurcations diagram that only displays endogenous fluctuations when increasing returns are considered. Hopf bifurcations may also occur under constant returns to scale. For this, the capital share of output must take a value between zero and 0.5, the elasticity of labor supply taking a positive value between zero and $+\infty$ (depending on the considered value for the capital share of output). As the degree of returns to scale increases from 1 to 2, the upper bounds for these ranges of admissible values fall to zero. Therefore, this fact does not shake the plausibility of nonlinear endogenous fluctuations, as long as returns to scale are not too increasing.[6] Indeed, as shown in a numerical example, Hopf bifurcations occur with reasonable values for the parameters considered in the model.

Some recent works, where more complicated frameworks are considered, have already shown that increasing returns to scale may contribute for the emergence of nonlinear fluctuations.[7] Therefore, the results of this paper are not overwhelmingly new, but this work has the advantage of clarifying some of the issues involved, once a simpler structure is considered. Bifurcation techniques can be easily handled and understood, by centering the discussion

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5. See, for instance, T. MACURDY [1981] and D. HUM and W. SIMPSON [1994].
6. As we shall see, in our model the relevant values for ρ are: $1 \leqslant \rho \leqslant 2$. The limit case of a duopoly is obtained with $\rho = 2$. Indeed, values for the degree of returns to scale higher than 2 are usually considered highly implausible. There are several studies showing evidence of increasing returns to scale in production. See, for instance, references in J. ROTEMBERG and M. WOODFORD [1993] and R. CABALLERO and R. LYONS [1990]. However, estimates point to values of the degree of returns to scale not much higher than 1.
7. See, for instance, G. CAZZAVILLAN, T. LLOYD-BRAGA and P. PINTUS [1998]. In that paper, a generalization of the Woodford model is considered with respect to two new factors: on one hand, there is capital-labor substitution and, on the other hand, there are increasing returns to scale. In view of footnote 4, if G. CAZZAVILLAN, T. LLOYD-BRAGA and P. PINTUS [1998] had considered no capital-labor substitution as a limit case, they would have obtained similar dynamics to those in the present paper. However, one should be aware that the introduction of capital-labor substitution in an OLG economy yields a different dynamic structure and results from those obtained by the introduction of capital-labor substitution in the Woodford model. On this, see G. CAZZAVILLAN and P. PINTUS [1997], and T. LLOYD-BRAGA [1995a], where OLG economies with capital-labor substitution and increasing returns are considered. See, also, M. ALOI, H. DIXON and T. LLOYD-BRAGA [1998] where increasing returns are introduced in a small open economy. These works, including the one presented in the current paper, follow the general lines of research considered in T. LLOYD-BRAGA [1995b].

on the values taken by relevant economic parameters.[8] It is frequently argued that nonlinear endogenous fluctuations only emerge under empirically implausible environments. Clearly, as also shown in this paper, this is far from a settled issue. More research in this area is required and extensions of the model presented to account for new and more natural assumptions, as for instance other imperfections in the market structure, are much welcome.

The paper proceeds as follows. In section 2, we present the details of the model and obtain the equilibrium dynamic system. In section 3, we analyze the occurrence of flip and Hopf bifurcations. Finally, in section 4, we present some concluding remarks.

2 The Model

We consider an overlapping generations economy with constant population, composed by a finite number of households living for two periods. In every period t, $t = 1, \ldots, \infty$, there are h identical young households, born in period t, and h old households born in $t-1$. Households work when young and consume when old, acting under perfect foresight.

The young household's income consists on wages, which are saved through asset accumulation to support consumption in the old age. We deal with two different specifications, depending on the asset considered: either productive capital or money. In the former case, young households invest wages in capital goods, k, which are rented to firms in the old age and fully depreciated in one period. In the latter, there is a fixed amount of money in the economy, M. Here, labor is the unique input and households save wages through money holdings. The equilibrium dynamics for this last specification can be directly obtained from the former, by assuming that capital is not needed for production. Therefore, we shall start by explaining the first case in some detail, while the second is naturally recovered as a special case.

There is a single (consumer and capital) good, with a price p taken as given by households. Let us consider that w denotes the competitive wage rate, r the competitive rental rate of capital, c consumption and l labor supply. Assuming a separable utility function, the problem solved by a representative young household born at $t \geqslant 1$ is written as follows:[9]

$$\max_{(k_t, c_{t+1}, l_t) \in \mathcal{R}_+^3} \{u(c_{t+1}) - v(l_t)\}, \text{ subject to:}^{10}$$

8. See J.-M. Grandmont, P. Pintus and R. de Vilder [1998]. They present geometrical methods that can be used in more complex models to get a simpler picture of how local bifurcations occur as a function of the "*parameters*" of the model.

9. We assume that the first generation in this economy, already born old at $t = 1$, is endowed with the total amount of capital, k_0, available in the economy at the outset of period $t = 1$, and spends all the rentals in consumption.

10. Note that, in (2), r_{t+1}/p_{t+1} is the expected real rental cost of capital services for the next period. Since we assume perfect foresight, at equilibrium its value is identical to that observed in period $t+1$.

$$(w_t/p_t)l_t = k_t \tag{1}$$

$$c_{t+1} = (r_{t+1}/p_{t+1})k_t \tag{2}$$

We assume that the utility function satisfies the following properties:

ASSUMPTION: u and v are continuous functions for $c \geqslant 0$, $0 \leqslant l \leqslant l^*$, where $l^* > 0$ is the workers' endowment of labor (possibly infinite). They are C^r, with r large enough, $u'(c) > 0$, $u''(c) \leqslant 0$ and $v'(l) > 0$, $v''(l) > 0$ for $c > 0$, $0 < l < l^*$, respectively.
Moreover, $\lim_{l \to 0} v'(l) = 0$ and $\lim_{l \to l^*} v'(l) = +\infty$.

Let $U(c)$ and $V(l)$ be defined as follows: $U(c) \equiv cu'(c) > 0$ for $c > 0$, and $V(l) \equiv lv'(l) > 0$, for $0 < l < l^*$. Then, the optimal solution $(k_t, c_{t+1}, l_t) \in \mathcal{R}^3_{++}$ of the household's decision problem, for each given vector $(w_t/p_t, r_{t+1}/p_{t+1}) \in \mathcal{R}^2_{++}$, must satisfy (1), (2) and the following equation:

$$U(c_{t+1}) = V(l_t) \tag{3}$$

Several identical firms produce the output, using a technology that exhibits (internal) increasing returns to scale. Accordingly, the technology is described by a production function homogeneous of degree $\rho \geqslant 1$, increasing and quasi concave in capital and labor inputs. Input markets are perfectly competitive. In each period t, every producer i maximizes profits, acting as a Cournot oligopolist under free entry and exit in the output market, identifying correctly the market demand function and taking as given aggregate income.[11] Using I_t to denote aggregate income, $I_t = hw_t l_t + hr_t k_{t-1}$, we obtain from (1) and (2) the following expression for the aggregate demand in the output market: $D_t = I_t/p_t = D(I_t, p_t)$[12]. Then, the profits' maximization problem solved by a representative producer i can be written as follows:[13]

$$\underset{y \in \mathcal{R}_{++}}{\text{Max}} \{py - C(y,w,r)\} \text{ subject to: } p = I/(Y_{-i} + y),$$

11. Hence, we assume that producers do not take into account "*Ford effects*", meaning that producers, while deciding the amount to produce, consider negligible any possible effect that their output may have on the demand for their products, *via* the income generated by them. See C. D'ASPREMONT, R. DOS SANTOS FERREIRA and L.-A. GÉRARD-VARET [1995], where "*Ford effects*" are considered. They present a model with *m* differentiated goods, where the productive sector is organized through a Cournotian monopolistic competition structure (producers take as given the total quantity of good produced by other firms and the prices of the other goods). One may notice that our assumptions on producers' behavior yield similar solutions and results to those we would obtain if we had adapted the Cournotian monopolistic structure to consider several differentiated goods (aggregated through the same CES index to yield the same composite good for capital and consumption). We just had to assume further that producers neglect any possible effect of their own decisions on the macro indicators (as aggregate income and output at the economy level and the general index price), which is a reasonable assumption if *m* is a large number.
12. The relevant (Marshallian) demand elasticity is then equal to 1.
13. In the next expressions, we ignore the index *t*, since all the variables refer to the same period.

where y is the amount produced by firm i and $C(y,w,r)$ is the cost function, increasing and homogeneous of degree $1/\rho$ in y.[14] The constraint represents the inverse of the demand function D, evaluated at the quantity Y of global production: $Y = y + Y_{-i}$, where Y_{-i} is the production of the other firms. Due to the Cournot assumption, Y_{-i} is taken as given by the firm i.

Since firms are all identical the natural equilibrium to consider is the symmetric one, where the n active firms in each period produce the same quantities y: $Y_{-i} + y = ny$. Hence, y is the solution of the following equation:

$$p(1 - 1/n) = \partial C(y,w,r)/\partial y, \tag{4}$$

where p is given by the constraint of the profits' maximization problem. This equation is the first order condition for the profits' maximization problem, evaluated at a symmetric equilibrium. It establishes the usual equality between marginal revenue and marginal cost.[15] However, producers are willing to enter into the market if they are able to make positive profits and, since there are no sunk costs, they will exit from the market if profits are negative. Therefore, ignoring integer constraints on the number of active firms, equilibrium will be attained when the number of active firms in the market is such that profits are zero, implying that $p = C(y,w,r)/y$. Once the cost function is homogeneous of degree $1/\rho$ in y, the following identity is verified: $\rho C(y,w,r)/y \equiv \partial C(y,w,r)/\partial y$. Using these facts and (4), we obtain the number of active firms at equilibrium in each period:[16]

$$n = \rho/\rho - 1, \tag{5}$$

where n, being uniquely determined by the exogenous parameter ρ, is constant over time. Moreover, the higher is the degree of returns to scale the lower is the number of active firms. When there are constant returns to scale, $\rho = 1$, the number of firms is infinite, leading by (4) to marginal cost pricing. Naturally, this case will be identified with the situation of perfect competition. When $\rho > 1$ there are increasing returns to scale and n is a positive finite number. Here, ρ is supposed to take values such that n is also an integer number. The duopoly case with $n = 2$ emerges when the degree of returns to scale takes the value 2. Therefore, we shall consider that $1 \leqslant \rho \leqslant 2$. In this model, the markup of prices over marginal cost, being identical to the degree of returns to scale, is constant with a value lower than 2.[17]

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14. Note that we do not consider the existence of fixed costs as the source of increasing returns. Here, average costs are decreasing along with decreasing marginal costs when $\rho > 1$.

15. Using similar procedures as in C. D'ASPREMONT, R. DOS SANTOS FERREIRA and L.-A. GÉRARD-VARET [1995], it can be shown that second order conditions are satisfied and that any Cournot equilibrium must be symmetric in quantities across the active firms.

16. The equilibrium solution considered here is indeed a subgame Nash perfect equilibrium of a two step game: in the first step, firms decide whether they enter into the market or not and, in the second step they choose the quantities to produce. See, for instance, FUDENBERG and TIROLE [1991].

17. Here, the markup is constant because both the market demand (Marshallian) elasticity and the number of firms are constant. Note that if increasing returns to scale were due to the existence of fixed costs, as in J. GALI and F. ZILIBOTTI [1995], then even with a constant demand elasticity, the number of firms, and therefore the markup, would be endogenously varying under free entry and exit.

As an exposition device, we assume that the number of households in each generation, *h*, is equal to the number *n* of active firms. Hence, equilibrium in the output market yields:

$$c_{t+1} + k_{t+1} = y_{t+1} \tag{6}$$

We consider that output is produced under a Leontief technology:

$$y_t = a[\min(l_t/z_l, k_{t-1}/z_k)]^\rho,\ a, z_l, z_k > 0, \tag{7}$$

where, at equilibrium in the input markets, l_t is the amount supplied by each young household at period *t*, and k_{t-1} is the amount of capital that each old household has available to rent at the outset of period *t*.[18] In the productive sector, cost minimization yields:

$$l_t/z_l = k_{t-1}/z_k \tag{8}$$

Finally, using (3), (6), (7), and (8) we can write the equilibrium dynamic equation of the model, in terms of labor, as follows:

$$U(c_{t+1}) = V(l_t), \quad \text{where:} \quad c_{t+1} = a(l_{t+1}/z_l)^\rho - (z_k/z_l)l_{t+2} \tag{9}$$

This equation defines a two-dimensional dynamic system, once it involves the second difference of l_t. The function $V(l)$ is invertible, since by assumption $V'(l) > 0$ for $0 < l < l^*$. Hence, fixing initial values for l_{t+1} and l_{t+2}, the dynamic equation (9) can be used to obtain equilibrium trajectories uniquely determined in the backward direction. However, notice that dynamics may not be uniquely determined in the forward direction if the function $U(c)$ is non monotonous.

A steady state solution (c,l), with $l_t = l_{t+1} = l_{t+2} = l$, satisfies the following equations:

$$U(c) = V(l), \quad \text{where} \quad c = a(l/z_l)^\rho - l(z_k/z_l) \tag{10}$$

We assume that (10) has a positive solution (c,l). Later on, we will show numerical examples where this assumption is satisfied.

For future reference, we now specify relevant expressions for the labor supply elasticity with respect to real wages, ε_{SS}, and for the capital share of output, SH_{SS}, both evaluated at a steady state solution:

$$\varepsilon_{SS} = \left(\frac{cU'(c)/U(c)}{lV'(l)/V(l) - cU'(c)/U(c)} \right) \tag{11}$$

$$SH_{SS} = 1 - (l/z_l)^{1-\rho}(z_k/a) \tag{12}$$

The expression (11) is obtained by differentiating both (3) and the budget constraint of the households: $c_{t+1} = (r_{t+1}/p_{t+1})(w_t/p_t)l_t$, while (12) follows from the use of (2), (6), (7) and (8). Note that ε_{SS} may take positive

18. Note that capital is a predetermined variable, *i.e.*, determined by past history. In period *t* the amount of capital available for production is fixed, having been determined by past savings.

or negative values, given the assumptions on the utility function. Note, also, that $SH_{SS} > 0$ for the positive values (c,l) satisfying (10).[19]

Lastly, let us consider the model's specification where money is the unique asset. Here, equilibrium is described through a unidimensional dynamic system, involving only one lagged variable:

$$(13) \qquad U(c_{t+1} = V(l_t), \quad \text{where:} \quad c_{t+1} = a(l_{t+1}/z_l)^{\rho}$$

As already mentioned, it can be directly recovered from (9) by considering $z_k = 0$. In this case, the production function (7) becomes $y_t = a(l_t/z_t)^{\rho}$, that is, capital is no longer needed for production. The household's problem is defined by the maximization of the same utility function subject to the following constraint: $c_{t+1} = (p_t/p_{t+1})(w_t/p_t)l_t$.[20] Given the existence of a fixed level of money M in the economy, the amount of *per capita* savings in real terms is now M/hp_t instead of k_t, and the real interest factor is now p_t/p_{t+1} instead of r_{t+1}/p_{t+1}. The first order conditions for the household's problem are still given by (3) together with the constraint above mentioned, and (11) still applies. Since the producers' cost function preserves homogeneity of degree $1/\rho$, the number of active firms is still given by (9).[21] However, once output is only used for consumption, equilibrium in the output market now reads as $c_{t+1} = y_{t+1}$, instead of (6). Equation (12) becomes irrelevant since there is no capital in this specification.

3 Flip and Hopf Bifurcations

In this section, we use local bifurcation analysis to detect the existence of equilibrium trajectories, satisfying (9) or (13), that exhibit endogenous fluctuations. Local Hopf and flip bifurcations are suitable mathematical instruments for our purpose.[22] Consider, for instance, the nonlinear dynamic

19. As we shall see in section 3, to study the occurrence of local bifurcations there is no need to fully parameterise functional forms for preferences and technologies. The analysis can generically be made using only economic relevant elasticities and shares evaluated at the steady state solution, whose values are susceptible of empirical evaluation.

20. This is the problem solved by a young household born at $t \geqslant 1$. It is assumed that old households alive in period $t = 1$ are endowed with the whole amount of money in the economy, which is used for consumption. Note, also, that p_{t+1} (or p_t/p_{t+1}, related to the inflation rate) in the constraint is the expected price of output for next period, which under perfect foresight is identical to that observed in period $t+1$. Equilibrium in each period requires $w_t/p_t = a(l_t/z_l)^{\rho-1}$, so that profits are zero. Hence, given observed values for l_t, c_{t+1} (and l_{t+1} by (13)) depend on the expected prices. If prices are expected to be constant, $p_{t+1} = p_t$, so is employment. Employment fluctuations are here caused by fluctuations in expectations about future prices. Similarly, the reader may note that, in the model with capital, fluctuations in employment are caused by fluctuations in expectations about the real rental cost of capital (or interest rate).

21. Note that the demand in the output market $(C = hc)$ is given by M/p, keeping unitary demand elasticity.

22. A reader interested in the details of bifurcation theory may wish to consult, for instance, J.-M. GRANDMONT [1988] or J. HALE and H. KOÇAK [1991].

system with two lagged variables: $y_{t+1} = g_1(y_t, x_t)$, $x_{t+1} = g_2(y_t, x_t)$. Let J be the Jacobian matrix of this dynamic system under consideration, and J_{SS} the matrix J evaluated at a steady state solution. A flip bifurcation generically occurs when, by continuously changing a parameter of the model, one of the eigenvalues of J_{SS} crosses the value –1, while a Hopf bifurcation generically occurs if a pair of complex conjugate eigenvalues of J_{SS} crosses the unit circle. Then, when parameters take values such that the eigenvalues of J_{SS} are close enough to the critical values above referred, the dynamic system has solutions defined by trajectories for the state variables that exhibit endogenous fluctuations. Along these trajectories the variables' values do not settle down but keep moving around the steady state within bounded intervals. Indeed, trajectories describing cycles of period two emerge when a flip bifurcation occurs, while under the occurrence of a Hopf bifurcation an invariant closed curve, where dynamics are periodic or quasi periodic, appears in the state space.[23]

3.1 Flip Bifurcations

We begin by considering the dynamic equation (13) for the model with money, where a steady solution (c, l) satisfies the following expressions: $U(c) = V(l)$, $c = a(l/z_l)^\rho$. Being an unidimensional dynamic model, only flip bifurcations are relevant as its Jacobian matrix has a single eigenvalue. One can easily check that this eigenvalue, at the steady state, is equal to –1 when: $\rho \dfrac{cU'(c)/U(c)}{lV'(l)/V(l)} = -1$. Given the assumptions on the utility function, and since ρ only takes positive values, this expression will be satisfied if, and only if, $U'(c) < 0$, implying by (11) a negative value for the labor supply elasticity. Indeed, from the above expression and using (11), we obtain the following bifurcation value for the elasticity of labor supply:

(14) $$\varepsilon_{SS}^F = -1/(1+\rho)$$

23. The trajectories exhibiting endogenous fluctuations emerge for parameter's values such that the eigenvalue is lower (higher) than –1 when the flip bifurcation is supercritical (subcritical). When the Hopf bifurcation is supercritical (subcritical) the trajectories exhibiting endogenous fluctuations emerge for parameter's values such that the modulus of the eigenvalues is higher (lower) than 1. In these cases of supercritical bifurcations, the orbits exhibiting endogenous fluctuations attract initial points sufficiently close to the steady state (*i.e.*, lying on the local unstable manifold). Note that the eigenvalues of J_{SS} are the eigenvalues of the linearized dynamics near the deterministic steady state. If the true system were in fact linear these trajectories instead of being attracted to the orbits exhibiting bounded fluctuations would become explosive. Indeed, in deterministic models as the one considered here, only nonlinearities are able to create the emergence of endogenous fluctuations (structurally stable). However, in linear models stochastic endogenous fluctuations (sunspots) close enough to the steady state may emerge if the steady state is indeterminate. The steady state is indeterminate when the number of stable eigenvalues is higher than the number of predetermined variables (as capital in our model). But this same phenomenon occurs in nonlinear models. Moreover, in nonlinear models if bifurcations are supercritical then there are also many stochastic equilibria (sunspots) around the deterministic cycles or around the invariant closed curve, respectively (see J.-M. GRANDMONT, P. PINTUS and R. DE VILDER [1998]). Whether bifurcations are subcritical or supercritical depends on higher derivatives of the dynamic system, usually hard to compute analytically. We will show whether one or the other case emerges by computer simulations for numerical examples.

Therefore, a flip bifurcation generically occurs when ε_{SS} crosses the negative value ε^F_{SS}. In this case, income effects on labor supply due to increases in real wages dominate substitution effects, the labor supply curve being negatively sloped at the steady state solution.

This result, that income effects should dominate substitution effects for the occurrence of a flip bifurcation, was already noticed in J.-M. GRANDMONT [1985], [1993] for models where markets are perfectly competitive. Let us explain why negative labor supply elasticity is indeed important for the existence of endogenous fluctuations, in this unidimensional dynamic model. To see this, consider an equilibrium orbit along which employment has increased for some period, $l_{t+1} > l_t$, and suppose that the labor supply elasticity is positive. Then, given the households' constraints, consumption also increases, $c_{t+2} > c_{t+1}$. Hence, at equilibrium, output must also increase and, by (13), employment increases again next period, $l_{t+2} > l_{t+1}$. Using again these same last arguments, we see that the employment sequence $(l_t, l_{t+1}, l_{t+2}, l_{t+3}, ...)$ is following an increasing monotonic trajectory and, thereby, endogenous fluctuations could not be observed.

By (14), the elasticity of labor supply must cross the value –0.5 when a flip bifurcation occurs under constant returns to scale (the perfect competition case). This is the condition one finds in the work of J.-M. GRANDMONT [1985]. And, as P. REICHLIN [1986] later pointed out: "*This may make endogenous cycles empirically implausible*". Indeed, empirical studies using micro data do not reveal such a strong negative value for the labor supply elasticity. However, as ε^F_{SS} is an increasing function of ρ, less negative values for the labor supply elasticity are required when increasing returns to scale are considered. Nevertheless, when ρ increases from 1 to 2, ε^F_{SS} only increases from (–1/2) to (–1/3), and it seems that (–1/3) is still a strong negative value for the labor supply elasticity of an average worker.

Notwithstanding, the existence of increasing returns to scale involves a new mechanism for the emergence of endogenous fluctuations through flip bifurcations. In fact, by constructing a bifurcation diagram over the parameter ρ, we shall see that it is possible to locate endogenous fluctuations for $\rho > 1$, fixing the others parameters of the model at values such that if ρ were equal to 1 those fluctuations would not emerge.

A Numerical Example

Let us consider the following utility function: $u(c_{t+1}) = z - (r/s)e^{(-sc_{t+1})}$ and $v(l_t) = l_t^b/b$, where $s, r > 0$ and $b > 1$.[24] The function $U(c)$, defined for $c \in (0, +\infty)$, is not invertible, and therefore dynamics are not uniquely determined in the forward direction. However, as already mentioned, the dynamic system (13) generates equilibrium trajectories uniquely determined in the backward direction, given initial values for l_{t+1}. Indeed, assuming $a = 1$ and $z_l = 1$, (13) may be written, in our example, as follows:

$$l_t = \left[r l_{t+1}^{\rho} e^{(-s l_{t+1}^{\rho})} \right]^{(1/b)} \qquad (15)$$

24. This utility function is identical to the one considered in A. MEDIO and G. GALLO [1992], chapter 12.

The expression (15) defines l_t as a function of l_{t+1}, and we use it to obtain equilibrium trajectories for employment defined in the backward direction.[25] This function has the familiar hump shaped configuration in the space (l_{t+1}, l_t), with a critical point $l^c = s^{-(1/\rho)}$. Under this configuration, a cascade of period doubling bifurcations usually occurs, susceptible of representation in a bifurcation diagram.[26]

A steady state solution *l* satisfies:

$$l^{b-\rho} = re^{(-sl^{\rho})} \tag{16}$$

Using (11), the condition (14) can be rewritten as follows:

$$\rho(1 - sl^{\rho})/b = -1, \tag{17}$$

where *l* satisfies (16).[27] Consider fixed values for *r*, *b* and *s*, and let l^* and ρ^* be the values of *l* and ρ such that (16) and (17) are satisfied. A flip bifurcation is then expected to occur when ρ crosses the value ρ^*.[28]

In order to obtain the bifurcation diagram presented in figure 1 proceed as follows. Fix $r = 6$, $b = 1.02$ and $s = 0.9$. Let the parameter ρ vary from 1 to 2 by steps of 0.01. Using (15), iterate backwards one thousand times its critical point l^c, for each value of ρ considered. Then, plot above each value of ρ the respective last hundred iterations. When ρ crosses from below the value ρ^*, a flip bifurcation occurs by which the steady state becomes unstable (in the backward dynamics) and a cycle of period two emerges.[29] Moreover, as ρ increases, a cascade of period doubling bifurcations occurs: a cycle of period 2 gives place to a cycle of period 4, followed by a cycle of period 8, etc. This bifurcation diagram shows the importance of increasing returns to scale for the emergence of endogenous fluctuations, since under constant returns to scale ($\rho = 1$) they would not appear.[30]

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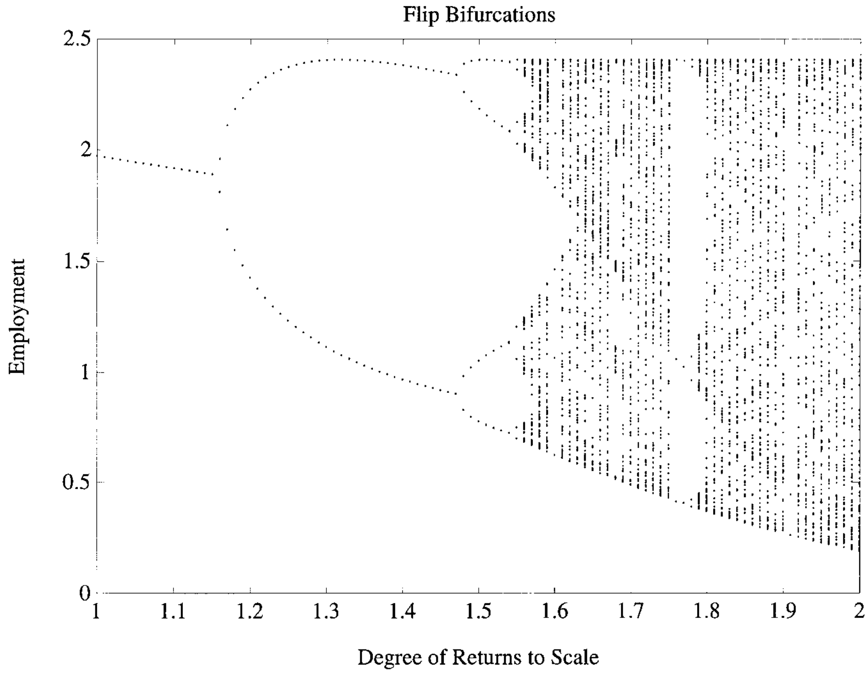
25. The analysis of backward dynamics is also relevant, even though time goes forward in real world. On one hand, all the periodic solutions of (15) obtained through backward analysis are also periodic solutions of the forward dynamics. And, on the other, if these solutions are stable in the backward dynamics (unstable in the forward direction) they become stable in the forward direction under learning rules, as it is shown in J.-M. GRANDMONT [1985].
26. For more details see, for instance, J.-M. GRANDMONT [1988], pp. 82-97. There, the reader may also see how to obtain the bifurcation diagram.
27. Note that it is always possible to choose a set of parameters *r*, ρ, *s* and *b* such that a steady state solution *l*, satisfying (16) and (17), exists. To calculate such a set of parameters one may proceed as follows: first choose values for $b > 1$, $1 < \rho < 2$ and $l > 0$. Then, determine the value for $s > 0$ such that (17) is satisfied. Finally, choose the value for $r > 0$ verifying (16).
28. Obviously, in our model, the relevant values of ρ are just those for which *n*, as given in (5), is an integer number. We are using ρ as the bifurcation parameter, instead of ε_{SS} as in the text, for the sake of our argument.
29. The flip bifurcation is subcritical, once in the real (forward) direction of time the steady state is stable (indeterminate) and the cycle is unstable.
30. The bifurcation diagram of figure 1 shows equilibrium trajectories for employment exhibiting bounded fluctuations. One should note that the equilibrium values of the other endogenous variables, like output, consumption and real wages, are linked to the values of *l* and, thereby, if employment fluctuates so do they.

FIGURE 1



3.2 **Hopf Bifurcations**

In this section, we consider the model with capital. Here we assume that $U(c)$ is an increasing function everywhere, that is: $U'(c) > 0$, for $c \in (0,+\infty)$. Therefore, (9) implicitly defines a dynamic function Ψ_1, engendering uniquely determined trajectories for l in the forward direction of time:

$$l_{t+2} = \Psi_1(l_{t+1},l_t) = (z_l/z_k)\{a(l_{t+1}/z_l)^{\rho} - U^{-1} \circ V(l_t)\} \tag{18}$$

Equation (18) can be rewritten as a system of two difference equations in two variables, by defining $g_{t+1} \equiv l_t$:

$$\begin{aligned} l_{t+2} &= \Psi_1(l_{t+1},g_{t+1}) \\ g_{t+2} &= \Psi_2(l_{t+1},g_{t+1}) = l_{t+1} \end{aligned} \tag{19}$$

Differentiating the system (19) we obtain the following Jacobian matrix, evaluated at a steady state solution (c,l):

$$J_{SS} = \begin{bmatrix} (\rho a/z_k)(l/z_l)^{\rho-1} & -z_l V'(l)/z_k U'(c) \\ 1 & 0 \end{bmatrix} \tag{20}$$

It is easy to check that a flip bifurcation is not a possible phenomenon for this dynamic system under the assumption that $U'(c) > 0$. In fact, the matrix J_{SS} can only have an eigenvalue equal to –1 if its trace and determinant satisfy the following relation: $-\det J_{SS} = 1 + \mathrm{tr}\, J_{SS}$. Using (20) this yields the following expression:

$$(21) \qquad -z_l V'(l)/z_k U'(c) = 1 + (\rho a/z_k)(l/z_l)^{\rho-1}$$

Once the r.h.s. of (21) is positive, and the l.h.s. is negative, (21) cannot be verified.[31]

However, the occurrence of Hopf bifurcations is possible with positive values for the labor supply elasticity. P. REICHLIN [1986] has shown this result, assuming perfect competition and constant returns to scale. Below, we show that Hopf bifurcations also occur with increasing returns to scale, as long as returns to scale are not too increasing.

The matrix J_{SS} has a pair of complex conjugate eigenvalues crossing the unit circle if its trace is inferior to 2 in absolute value when its determinant is crossing the value 1. Using (10), (11) and (12) we obtain the critical value for the labor supply elasticity, ε_{SS}^{H}, so that $\det J_{SS} = 1$:

$$(22) \qquad \varepsilon_{SS}^{H} = SH_{SS}/(1 - 2SH_{SS}),$$

and an upper bound for the capital share of output, SH_{SS}, so that $|\mathrm{tr}\, J_{SS}| < 2$:

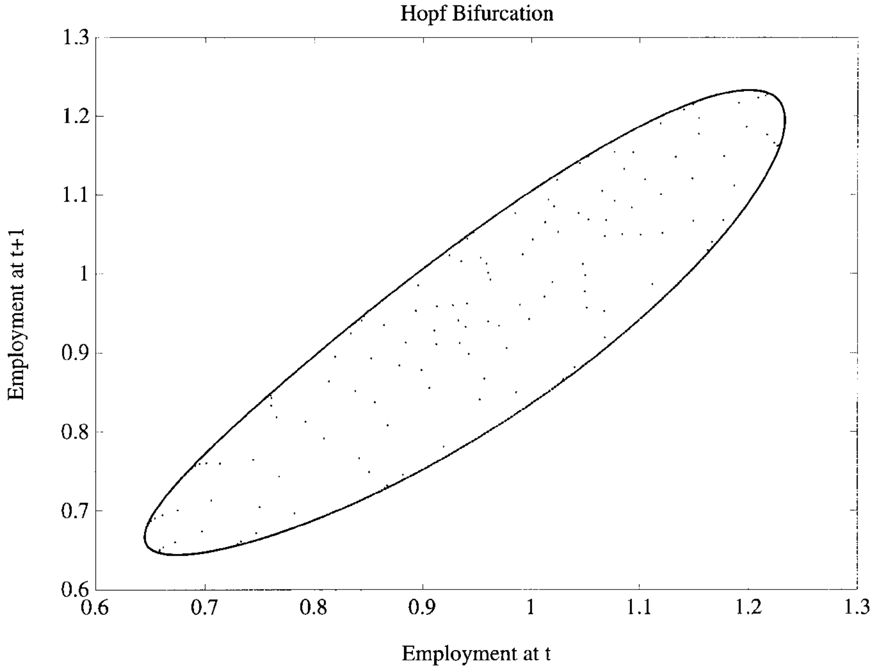
$$(23) \qquad SH_{SS} < 1 - (\rho/2)$$

Note that the labor supply elasticity must be positive at the steady state whenever both (22) and (23) are verified, since $\rho \geqslant 1$ and $SH_{SS} > 0$. A Hopf bifurcation generically occurs when ε_{SS} crosses the value ε_{SS}^{H}, provided that SH_{SS} and ρ satisfy (23). The condition obtained in P. REICHLIN [1986], that the capital share of output must be inferior to 0.5 for a Hopf bifurcation to occur, is here recovered by considering $\rho = 1$.

Expressions (22) and (23) suggest that the admissible upper bound for ε_{SS}^{H} decreases as ρ increases. The reason for this inverse relation may be explained as follows. Let us suppose that the elasticity of labor supply is positive, a necessary condition for a Hopf bifurcation. Consider an equilibrium trajectory along which employment is growing at an increasing rate of growth for some periods. Given the Leontief technology, employment will increase even more if capital increases. Yet, if the considered trajectory exhibits endogenous fluctuations, the employment growth must reverse at some period and therefore capital should not increase too much. Given a rate of growth for employment, the higher is ρ the higher is the increase in the output. In equilibrium, output is equal to consumption plus capital (since investment in this model with total depreciation is equal to capital). Hence, if capital does not increase too much, the higher is ρ the more should consumption increase. But, given a fixed increase in employment l_t, a higher increase in consumption is obtained with a lower elasticity of labor supply, as can be seen using (1) and (2).

31. Indeed, using (10), (11), (12) and (21), one may notice that the occurrence of a flip bifurcation requires again a negative value for the labor supply elasticity, but the higher is the degree of returns to scale the less negative is the required value for the labor supply elasticity.

FIGURE 2



Indeed, using (22) and (23), we have that: $\varepsilon_{SS}^{H} < (2-\rho)/2(\rho-1)$. Hence, while under constant returns to scale ε_{SS}^{H} may take any positive value, the upper bound for this range decreases to zero as ρ increases from 1 to 2. However, this fact does not interfere with the plausibility of Hopf bifurcations, since the majority of estimates for ε_{SS} point to positive but low values and estimates for ρ point to values lower than 2. By condition (23), we may, also, notice that the admissible upper bound for the capital share of output decreases from 0.5 to zero as ρ increases from 1 to 2. The capital share of output in the industrialized countries is a number somewhere between 0.25 and 0.4. Thereby, if the degree of returns to scale is not too high, namely ρ not greater than 4/3, Hopf bifurcations are compatible with plausible values for the capital share of output. This upper bound on ρ is not restrictive since empirical works point to values for ρ not much higher than 1.

In conclusion, the emergence of endogenous fluctuations through Hopf bifurcations is possible under empirically plausible values for the capital share of output, for the labor supply elasticity and for the degree of returns to scale. Below, we show a numerical example of this fact, where a stable invariant closed curves emerges with $\rho = 1.2$.

A Numerical Example

We consider the following utility function: $u(c_{t+1}) = c_{t+1}^{\alpha}/\alpha$, where $0 < \alpha < 1$ and $v(l_t) = l_t^b/b$, where $b > 1$.

The production function is given by (7), and we assume that $a > 1$, $z_l = 1$ and $z_k = a - 1$. The dynamic function (18) can then be rewritten as follows:

(24) $$l_{t+1} = (al_{t+1}^{\rho} - l_t^{b/\alpha})/(a-1),$$

where $l = 1$ is always a steady state solution. At this steady state solution the capital share of output, given by (12), is identical to: $SH_{SS} = 1/a$. Using (11) we obtain a constant positive value for the labor supply elasticity: $\varepsilon_{SS} = [(b/\alpha) - 1]^{-1}$.

In our example, we fixed $a = 3$. Hence, $SH_{SS} = 1/3$. In this case, a Hopf bifurcation occurs when ε_{SS} crosses the value 1, as can be seen using (22). Figure 2 is obtained assuming that $b/\alpha = 2.1$, that is, $\varepsilon_{SS} = 1/1.1$. We, also, considered that: $\rho = 1.2$. The eigenvalues of the Jacobian matrix evaluated at $l = 1$ are then complex conjugate numbers with modulus slightly higher than one: the steady state is unstable. We started with an initial point for (l_t, l_{t+1}) near (1,1), and using (24) we obtained a trajectory with three thousand elements. Then, we plotted it in the state space (l_t, l_{t+1}). As we can observe in figure 2, the equilibrium trajectory of l fluctuates forever, converging to an invariant closed curve that surrounds the repelling steady state.[32]

4 Concluding Remarks

In this paper, we develop a simple overlapping generations model to study the impact of increasing returns to scale on the emergence of nonlinear endogenous fluctuations. We introduce increasing returns to scale at the firm level in a Leontief technology, and assume the existence of Cournot competition with free entry and exit in the output market. The models considered in J.-M. GRANDMONT [1985], P. REICHLIN [1986], R. DE VILDER [1996], and WOODFORD [1986], all of them assuming the existence of constant returns to scale and perfect competition in the output market, can be recovered as limit cases of our model.

The paper has also a pedagogical purpose, as the required mathematics to workout the results are easy to manage in the simple framework considered. The basic steps in our procedure can then be easily followed. We analyzed the occurrence of local flip and Hopf bifurcations in order to study the emergence of nonlinear endogenous fluctuations. We established simple conditions under which local flip and Hopf bifurcations generically occur. These conditions are established in terms of the relevant "*parameters*" of the model, whose values may be subject to empirical evaluation: the degree of returns to scale, the elasticity of labor supply and the capital share of output evaluated at the steady state solution. We compared the required values for these parameters, so that flip and Hopf bifurcations occur, to their usual values obtained in empirical estimations.

Through simple computer experiments, we also simulated the occurrence of these bifurcations and the emergence of nonlinear endogenous fluctuations,

32. The Hopf bifurcation is supercritical. Note that if we ignored the existence of local nonlinearities, we would not be able to identify the existence of endogenous fluctuations since every trajectory, except the (determinate) steady state itself, would become explosive under the parameterization considered.

assuming reasonable values for the "*parameters*". We have shown an example where under increasing returns cycles of period two emerge, through the occurrence of a flip bifurcation, but do not emerge under constant returns to scale, considering fixed values for the other parameters of the model. We have also shown an example where an invariant closed curve emerges, through Hopf bifurcations, considering plausible low levels of increasing returns to scale, and reasonable values for the labor supply elasticity and for the capital share of output.

Overall, our results show that, by taking the existence of increasing returns as a fact, the environment under which endogenous fluctuations can emerge approaches that usually considered as being empirically plausible.

The importance of increasing returns to scale for the emergence of nonlinear endogenous fluctuations has already been emphasized in other recent works, where more complicated frameworks are considered. See, for instance, G. CAZZAVILLAN, T. LLOYD-BRAGA and P. PINTUS [1998], T. LLOYD-BRAGA [1995] and G. CAZZAVILLAN and P. PINTUS [1997]. These works, by developing the M. WOODFORD [1986] model or the overlapping generations model here presented to account for capital-labor substitution, show that increasing returns to scale are important for the occurrence of endogenous fluctuations if capital and labor are not highly complementary in production.

To understand whether the emergence of nonlinear endogenous fluctuations is a mere theoretical curiosity or whether their emergence is, on the contrary, a possibility under plausible environments, further developments of the model here presented are also desired. Promising extensions, under current research, are the consideration of markup variability or imperfect competition in the labor market in addition to increasing returns to scale and capital-labor substitution in production. However, many other possible directions for further research are still open, like the consideration of other capital market imperfections or of open economies. We, strongly, believe that the introduction of increasingly realistic assumptions on market structure will bring new interesting results on the emergence of nonlinear endogenous fluctuations.

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