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On Knowledge Diffusion, Patents Lifetime and Innovation Based Endogenous Growth

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ABSTRACT. – This paper analyzes the macroeconomic effects of the patents system within the framework of an endogenous growth model with new products development. We assume that patents not only represent a commercial protection for innovators but also entail a partial property right on information. Therefore, increasing the patents lifetime increases the profitability of a given research and development project but also decreases the knowledge spillovers that play a crucial role in the growth process. We then show that when the instantaneous diffusion of knowledge is "low", growth is maximized by a finite patents lifetime while this role is devoted to infinitely lived patents are growth-maximizing when the instantaneous diffusion of knowledge is "high". Furthermore, in the former case, the optimal patents lifetime is also finite and shorter than the growth maximizing one. The design of an optimal patents policy only holds in a second best analysis. When the resource allocation is determined by a central planner maximizing the utility of a representative agent, social welfare is always higher than in the decentralized case.

Diffusion du savoir, durée de vie des brevets et croissance endogène

RÉSUMÉ. – Cet article propose une étude des effets macro-économiques du système de brevets dans le cadre d'un modèle de croissance endogène avec création de produits nouveaux. Nous supposons que les brevets non seulement procurent une protection commerciale pour les innovateurs mais entraînent également un droit de propriété partiel sur l'information. Dans ces conditions, une augmentation de la durée de vie des brevets augmente la rentabilité d'un projet de recherche et développement donné, mais réduit les surplus de connaissance qui jouent un rôle crucial dans le processus de croissance. Nous montrons que lorsque la diffusion instantanée de la croissance incorporée à une innovation protégée est « faible », la croissance est maximisée par une durée de vie finie pour les brevets, alors que cette durée de vie est infinie si la diffusion est « forte ». De plus, dans le premier cas, la durée de vie optimale des brevets est finie et plus courte que celle qui maximise la croissance.

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1 Introduction

The microeconomics of innovation that underly the growth process is now a strong line of research within the field of endogenous growth theory. The canonical models – AGHION and HOWITT [1992], GROSSMAN and HELPMAN [1991a, b], ROMER [1990], SEGERSTROM *et al.* [1990] – share a common but never fully justified hypothesis: patents are infinitely lived. In reality, this hypohesis is evidently not verified (the statutory lifetime is 20 years in France, 17 in the US) and it would obviously not be optimal to set patents lifetime to infinity since this would create permanent monopolies with a negative effect on welfare. As an industrial policy tool, patents always result from an arbitrage between maintaining private incentives for innovative investment and limiting the market distortions induced by monopolies. In this paper, we try to understand why this kind of arbitrage never appears in the canonical models, we detail the nature of patents and then we study the impact of their lifetime on growth and welfare.

Actually, the main reason which justifies the "canonical hypothesis" is the fact that for the purpose of modeling, setting an infinite patents lifetime is the most simple way to proceed. Indeed, under finite patents lifetime, two difficulties appear. First, the dynamics of the model are described by non-linear delayed differential equations whose study is far from being completely described by mathematicians and is in many cases, at this time, impossible to conduct. Second, the price vector never reflects the cost structure and the consumers' purchase are not efficiently spread over the different goods. Indeed, households face two kinds of goods: the competitive ones – that are no longer protected by a patent and are priced at marginal cost – and the monopoly ones whose price includes a positive markup over marginal cost. Obviously, infinitely lived patents do not systematically guarantee an optimal sharing out of purchases, but the problem is always simpler when production is devoted to monopolies only. This argument becomes even more obvious for the "expanding variety" models, in the tradition of GROSSMAN and HELPMAN (1991a, ch. 3). Indeed, under infinitely lived patents when (i) the utility function or the final goods' production function is CES – in the way of DIXIT and STIGLITZ [1977] –, uses only differentiated goods as arguments and treats them symmetrically, and when (ii) at the initial state of the model, there does not exist any previously invented good whose production rights are in the public domain, the production of the horizontally differentiated goods is devoted to monopolies only who all adopt the same markup over marginal cost. Under these conditions, the price structure reflects exactly the cost structure and since the positive profits are distributed to the households, the presence of monopolies does not entail any net distortion [1]. The only sub-optimality of the model comes from the public good nature of knowledge and since its diffusion is not affected by variations in the patents lifetime, it is always optimal to set it to infinity.

1. See GROSSMAN and HELPMAN [1991a], p. 70f.

At this point, we see that in order to reconcile the innovation based endogenous growth literature with reality in terms of patents lifetime, we miss a negative effect of too long a patents lifetime on welfare. Since we cannot expect this negative effect to come from the existence of permanent monopolies, we argue that it may come from the knowledge diffusion process. We believe that the conception of patents, used in canonical models, is in some sense too radical since it makes the level of public knowledge independent of patents lifetime.

Indeed, it is generally admitted that a patent is a contract between an inventor (firm or individual) and the government. This contract is designed in order to ensure a maximal diffusion of knowledge. The government guarantees commercial protection to the inventor in exchange for a detailed diffusion of the corresponding technological information. This information must appear in a form [2] that is made public by the patents bureau within the eighteen months – in the case of France – that follow the date of the patent demand. Thus, taking this idea in a strictly literal fashion, the canonical literature assumes that the whole stock of technological knowledge is public.

But many jurists consider that the information diffused by the patents bureau during the protection period does not represent the whole technological knowledge attached to an invention. In France, it is generally admitted that at most 80% of the technological information appears in the demand form [3]. In order to capture this idea, and following ARROW [1994], we think that despite the fact that knowledge has many of a public good's attributes, all knowledge generated by an economy does not spread instantaneously. Secrecy can be kept on some technological information relative to the private production of a given good. Production experience, learning-by-doing, know-how, are elements that constitute a tacit knowledge, not codified, that does not spread as quickly as technological knowledge in its wide meaning. Furthermore, it is clear that a share of codifiable knowledge may also be kept secret for strategic reasons.

The key point of our analysis is then the assumption that only a share of the knowledge developed through private research and development (R&D) activities diffuses instantaneously, the complementary (private) share diffusing only when the patent protection ends. As long as a prdoucer keeps its monopoly right on the production of a given good, it keeps a given quantity of private knowledge. This is due to two kind of things: first to the fact that a part of this knowledge cannot be codified in a way allowing its use by competitors and secondly to intentional secrecy. But when the patent protection ceases, anybody can produce the good and then has access to the whole knowledge associated with it. Therefore the amount of public knowledge depends on the patents lifetime. Under infinitely lived patents, the amount of public knowledge may be too low to be optimal and it may then become welfare improving – despite the price distortion that will

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2. The law requires that a patent application form must contain among others: a description of the technical domain, a description of the prior "state of the art", a detailed exposition of the invention, of the technical difficulties, of the technical solutions, a detailed exposition of at least one way to realize the invention, a list of the possible industrial uses, etc.

3. This value is given by the Institut National de la Propriété Industrielle.

appear – to shorten the patents lifetime in order to reduces the share of private knowledge. This will allow us to obtain the negative effect of too long a patents lifetime that canonical models lack. As suggested by DRÈZE [1980], the government can artificially modify knowledge excludability in order to reach a second best optimum [4].

In this paper, in order to study the optimal lifetime of patents, we construct a model based on JUDD [1985] and on GROSSMAN and HELPMAN [1991a]'s expanding variety endogenous growth model. These models are very similar in their basic structure. The main difference is the fact that Judd's model is an exogenous growth model where R&D activities do not produce any knowledge spillover. Innovation is endogenous, but not sustainable when there is no exogenously increasing factor (such as an increasing labor force). The absence of knowledge spillovers can be interpreted as the fact that the knowledge created through the invention of a given variety is product-specific (*i.e.* cannot be used to develop a new variety) and is the private property of the inventor during the patent protection period. This knowledge becomes public when the patent protection ends, but is used only by the competitive producers of the corresponding variety. Thus, Judd's paper does not take into account the effect of horizontal knowledge diffusion. In this framework, given that the nature of knowledge does not entail any sub-optimality, Judd demonstrates that when the initial variety degree is nil, the first best patents policy is to set lifetime to infinity for reasons similar to the ones that make infinittely lived patents the second best solution in GROSSMAN and HELPMAN [1991, ch. 3].

At the opposite extreme, Grossman and Helpman consider that knowledge created through R&D activities is both non rival and non excludable, and can be used in any product line. In short, this means that in the Grossman and Helpman's model, a patent is only a production right that protects against competition (and imitation) but does not preclude the use of knowledge by other agents. This hypothesis, which makes innovation and growth sustainable – as long as the spillover is specified in order to ensures constant returns in the R&D process – explains why, as we have already noted, the patents lifetime has no effect on the diffusion of knowledge.

Our model is closer to Grossman and Helpman's than to Judd's in the sense that it is a model of *endogenous growth* where we have specified the knowledge diffusion process in order to guarantees constant returns to R&D activities.

There does not exist a large macroeconomic literature that studies the incentives and the distortions resulting from a patents system. NORDHAUS [1969] is the first one to study the question in terms of economic policy, but his work is done in a static framework that cannot be used in order to evaluate the dynamic efficiency – *i.e.* the rythm and the rentability of intertemporal investment – of a patents system. A major study, and to date the only one developed in a dynamic general equilibrium framework is JUDD

4. Obviously, the design of an optimal patents policy makes sense only if the government canot directly control the allocation of resources. If the government could do so, the aggregate investment would no longer be decided by profit maximizing agents.

[1985] whose main characteristics have just been noted. More recently, CHOU and SHY [1991] studied the consequences of patents lifetime on the equilibrium innovation path using different specifications for the R&D cost function, but their analysis is conducted in a partial equilibrium framework. Finally, GUELLEC and RALLE [1993] modify the Grossman and Helpman's variety model by assuming a finite patents lifetime, but limit the study to an approximation of the innovation rate expressed as a linear increasing function of the patents lifetime when the latter is "relatively short".

The present paper contains three major contributions to the innovation based endogenous growth literature. First, it introduces a new hypothesis that results from a better analysis the role of knowledge as an input of the R&D process. Second, it provides a full characterization of the steady state innovation path in a general equilibrium model of endogenous growth with finite patents lifetime. This characterization allows us to distinguish the situations where the optimal patents lifetime is finite and the ones where it is infinity. Third, it shows that in general, the optimal patents lifetime is not innovation – and growth – maximizing. Furthermore, one can also intepret our model as a generalization of Grossman and Helpman's expanding variety model, the latter then representing a useful benchmark economy with respect to which we can evaluate the effects of restrictions in knowledge diffusion.

The discussion is developed as follows. Section 2 states the basis of the model, section 3 establishes two different specifications of the patents system, section 4 studies the welfare properties of the patents system and section 5 concludes.

2 The Model

There are two kinds of agents: the households that supply inelastically the constant labor force L and the firms who produce a set of horizontally differentiated goods (that we also call "varieties") with labor as the only input. Innovation is modelled as introducing a new variety into the economy. This process is costly since it requires an investment of resources into R&D. The firm that engages such R&D expenses can produce a new variety and is protected from competition for a given length of time T that corresponds to a patents lifetime. The positive profits earned, thanks to this local monopoly position, are used to repay the cost of R&D. Once the patent protection has stopped, the corresponding variety falls into the public domain and is therefore produced by competitive firms. The households have a preference for diversity and growth is measured in terms of utility.

2.1. The Households

The representative (price taker) household maximizes:

(1) $$U = \int_0^{\infty} e^{-\rho t} \ln D(t)\, dt$$

where ρ is the subjective discount rate and $D(t)$ is a static utility index inspired from DIXIT and STIGLITZ [1977].

(2) $$D(t) = \left(\int_0^{n(t)} x(i, t)^{\alpha} \, di \right)^{\frac{1}{\alpha}}$$

In this equation, $n(t)$ is the "number" of varieties [5] available in t, $x(i, t)$ is the quantity consumed in t of variety i and $\alpha = 1 - \sigma^{-1}$. The parameter $\sigma > 1$ stands for the elasticity of substitution between two varieties. Defining $A(t)$ as private wealth, the intertemporal budget constraint is:

(3) $$\begin{cases} \dot{A}(t) = r(t) A(t) + w(t) L - \int_0^{n(t)} p(i, t) x(i, t) \, di \\ \lim\limits_{t \to \infty} R(0, t) A(t) \geq 0 \end{cases}$$

where $r(t)$ is the interest rate, $w(t)$ the wage rate, $p(i, t)$ the price of variety i, and

$$R(t_0, t) = \exp\left[- \int_{t_0}^{t} r(\tau) \, d\tau \right].$$

The representative household's program can be decomposed into a static one – concerning the allocation of consumption among varieties – and a dynamic one – concerning the determination of savings. The resolution of the former leads to the following demand functions [6]

(4) $$x(i) = \frac{E p(i)^{-\sigma}}{\int_0^n p(j)^{1-\sigma} \, dj} \quad i \in [0, n]$$

where E represents consumption expenditures. In order to solve the second program, we define a price index P such that $E = PD$.

$$P = \left(\int_0^n p(i)^{1-\sigma} \, di \right)^{\frac{1}{1-\sigma}}$$

We can then substitute E/P for D in the utility function (1) and E for $\int_0^n p(i) x(i) \, di$ in the budget constraint (3). Given that the representative household is a price taker, his optimal consumption expenditures path is defined by:

(5) $$\begin{cases} \dot{E}(t) = E(t) [r(t) - \rho] \\ \lim\limits_{t \to \infty} R(0, t) A(t) = 0 \end{cases}$$

5. Formally, n is the measure of the continous set of varieties. In the following, for the sake of simplicity, we shall speak of "number" of varieties, just as if they were constituting a discrete set.

6. In the following, we drop the time index and the variety index when they are not strictly necessary for comprehension.

2.2. **The Firms**

We have two kinds of firms: the competitive ones, acting in branches 0 to n_c, and the monopolist ones, producing varieties n_c to n. They are all characterized by the same production function $x(i) = l(i)$ where $l(i)$ is the amount of labor used in branch i. The monopolistic firms, protected by a patent, maximize their profit $\pi(i) = [p(i) - w]\, x(i)$, $i \in]n_c, n]$, given the demand function (4). The resulting strategic price and profit are:

$$p(i) = \frac{w}{\alpha} = p_m \quad i \in]n_c, n] \tag{6}$$

which implies by (4) $x(i) = x_m$ for $i \in]n_c, n]$, and

$$\pi(i) = \frac{1-\alpha}{\alpha} w x_m = \pi \quad i \in]n_c, n]. \tag{7}$$

The competitive firms charge a price equal to marginal cost and earn no profit.

$$p(i) = w = p_c \quad i \in [0, n_c] \tag{8}$$

The market value of a monopolistic firm created at date t, $v^e(t)$, is equal to the present value of the flow of profit it will earn until the end of the patent protection.

$$v^e(t) = \int_t^{t+T} R(t, z)\, \pi(z)\, dz \tag{9}$$

A potential "innovator" accepts to introduce a new variety only if the cost of entry – *i.e.* the cost of the R&D expenses $v(t)$ – is not larger than $v^e(t)$. Since we assume that R&D is a free entry activity, no profit opportunity is left unexploited, so that we always have $v^e \leq v$. When the R&D cost is greater than the market value $(v^e < v)$, there is no innovation, and no growth. When $v^e = v$, potential innovators are indifferent between investing or not and the aggregate entry level is determined by the availability of the labor force.

2.3. **R&D**

We now turn to the important point of this paper: the determination of the R&D cost, $v(t)$. Following GROSSMAN and HELPMAN [1991, ch. 3], we assume that inventing a new variety is an activity that requires labor and public knowledge, and has constant returns with respect to labor (the only "private" input). An invention can be considered as producing knowledge, whose aggregate stock is designated by $n(t)$. As mentioned in the introduction, this knowledge is non rival and, thanks to the patents system, partially excludable. Therefore, the stock of public knowledge at date t – *i.e.* the stock of knowledge freely available for anybody – is composed of $n_c(t)$, the whole knowledge developed through the invention of varieties that are in the public domain at date t, and a share θ of the knowledge associated with the invention of varieties still protected by a patent at the same date. Thus, defining $\Delta_n(t) = n(t) - n_c(t)$, the stock of public knowledge available at date t is:

$$k_n(t) = n_c(t) + \theta\Delta_n(t) \quad 0 \leq \theta \leq 1. \tag{10}$$

Public knowledge increases the productivity of the labor force involved in R&D activities. In order to obtain a constant growth rate, we assume that invention – *i.e.* production of new knowledge $\dot{n}$ – is a linear function of the stock of public knowledge (k_n). We can then define the following "production function" for R&D:

$$\dot{n}(t) = \frac{1}{a} k_n(t) L_n(t) \tag{11}$$

where $a > 0$ is a productivity parameter and L_n is the labor devoted to R&D. Therefore, the cost of developing and introducing a new variety, or equivalently the cost of a patent, can be stated as follows:

$$v(t) = \frac{aw(t)}{n_c(t) + \theta\Delta_n(t)}; \tag{12}$$

When θ, that we call the *diffusion coefficient*, is equal to one – *i.e.* when knowledge is completely non excludable despite the presence of a patent – and when $n_c = 0$ – *i.e.* when patents are infinitely lived with $n_c(0) = 0$ – the specification of the R&D activity is similar to GROSSMAN and HELPMAN's. JUDD [1985]'s specification corresponds to the particular case $k_n = 1$. This hypothesis means that knowledge is always a private good (that disappears when the patent protection ends) and has no effect on the R&D labor productivity. Due to this hypothesis, Judd's model never displays endogenous growth. Indeed, the market value of a monopoly decreases over time with the increase in the number of varieties, and the rentability of R&D investments is therefore also decreasing. As in more usual neoclassical growth models, the growth rate tends to zero if no exogenous force (as an increase in the labor force) is maintaining it. Assuming that the cost of a patent is decreasing with the number of varieties allows for sustained growth. In Grossman and Helpman's model, v is assumed to be a decreasing function of the instantaneous number of varieties. In our model, v decreases proportionally to a lagged number of varieties.

The diffusion coefficient, θ, is exogenous. This means that agents cannot control the part of the knowledge that diffuses instantaneously. In the model, such an hypothesis does not represent any problem since θ determines the sharing-out of something that, in each case, is considered as an externality. In reality, this hypothesis means that agents cannot decide to deliberately maintain secrecy concerning the scientific discoveries that are incorporated into their currently produced goods. This can be considered as a restrictive assumption, but it is necessary to display relatively simple results and it is the only one consistent with the idea of a continuum of innovators/producers.

2.4. **Closing the Model**

The model is closed by a resource constraint that specifies the use of the labor force. Labor demand from R&D, L_n, is given by (11).

$$L_n = a\,\frac{\dot{n}}{k_n} \tag{13}$$

Labor demand from the production sector is given by $L_x = n_c\,x_c + \Delta_n\,x_m$. Using (4), (6) and (8), we have:

$$x_c = \frac{E/w}{n_c + \alpha^{\sigma-1}\,\Delta_n} \qquad \text{and} \qquad x_m = \alpha^{\sigma}\,x_c < x_c. \tag{14}$$

The discrepancy between the quantity consumed of competitive varieties and the quantity consumed of monopolistic varieties results from the price distortion due to monopolies. The resource constraint is given by:

$$L = \frac{E}{w}\left(\frac{n_c + \alpha^{\sigma}\,\Delta_n}{n_c + \alpha^{\sigma-1}\,\Delta_n}\right) + a\,\frac{\dot{n}}{k_n} \tag{15}$$

In equation (15), the value of the terms between parentheses belongs to $[\alpha,\,1]$ and tends to 1 when the share of competitive varieties is large. Since all varieties are produced using the same quantity of labor, everything being equal elsewhere, the less the share of monopolies, the higher the aggregate labor demand from the production sector.

We adopt here the normalization proposed by Grossman and Helpman which consists of setting to one the value of the consumption expenditures. This normalization is useful since it provides a very simple expression for the interest rate. Indeed, with $E\,(t) \equiv 1$ and (5), the nominal interest rate is just equal to the subjective discount rate: $r\,(t) = \rho$. It is then variations of the wage rate w that ensure equilibrium on the labor market.

The dynamics of the model are then represented by

$$\dot{n}\,(t) = \frac{n_c\,(t) + \theta\Delta_n\,(t)}{a}\left[L - \frac{1}{w\,(t)}\left(\frac{n_c\,(t) + \alpha^{\sigma}\,\Delta_n\,(t)}{n_c\,(t) + \alpha^{\sigma-1}\,\Delta_n\,(t)}\right)\right] \tag{16}$$

where $n_c\,(t) = n\,(t-T)$. Defining

$$\tilde{w} = \frac{1}{L}\left(\frac{n_c + \alpha^{\sigma}\,\Delta_n}{n_c + \alpha^{\sigma-1}\,\Delta_n}\right) \tag{17}$$

we have

$$\begin{cases} \text{If } v^e\,(t) \leq \dfrac{a\tilde{w}\,(t)}{n_c\,(t) + \theta\Delta_n\,(t)} & \dot{n}\,(t) = 0 \text{ and } w\,(t) = \tilde{w}\,(t), \\ \text{If not, } v^e\,(t) = \dfrac{aw\,(t)}{n_c\,(t) + \theta\Delta_n\,(t)} & \dot{n}\,(t) > 0 \text{ and } w\,(t) > \tilde{w}\,(t) \end{cases} \tag{18}$$

where, by (7) and (14):

$$v^e(t) = (1-\alpha)\int_t^{t+T} \frac{e^{-\rho(z-t)}}{\alpha^{1-\sigma}\, n_c(z) + \Delta_n(z)}\, dz \tag{19}$$

In the case $\dot{n} > 0$, the equilibrium value of the wage rate w is determined forwardly and equation (16) is a delayed differential equation depending on $n_c(t)$, with $n_c(t) = n(t-T)$.

We now study the solutions of this model with infinitely lived patents and then with finite patents lifetime.

3 Different Specifications of the Patents System

Innovation based endogenous growth models have the specific property that innovation (and growth) is possible only if the labor supply is high enough to allow both production and R&D activities. The long run equilibrium of the model is represented by a steady state which can be purely static – if the labor supply is too scarce – or can be characterized by a constant and positive growth rate. In the following, we study successively these two possibilities.

3.1. Infinitely Lived Patents

The case with infinitely lived patents corresponds formally to $T = \infty$, $n_c = 0$ and $k_n = \theta n$. This is the simplest case to study since there is no "competitive variety" and therefore no price distortion on the differentiated goods market. The dynamics of the model in this case are described by:

$$\begin{cases} \text{If } v^e(t) \le \dfrac{a\tilde{w}}{\theta n(t)}, & \dot{n}(t) = 0 \text{ and } w(t) = \tilde{w}, \\ \text{If not, } v^e(t) = \dfrac{a w(t)}{\theta n(t)}, & \dot{n}(t) = \dfrac{\theta n(t)}{a}\left(L - \dfrac{\alpha}{w(t)}\right), \end{cases} \tag{20}$$

where $\tilde{w} = \alpha / L$ and

$$v^e(t) = (1-\alpha)\int_t^{\infty} \frac{e^{-\rho(z-t)}}{n(z)}\, dz.$$

Solutions without growth

In this case, the total number of varieties is constant: $n(t) \equiv n_0$. The value of a monopoly is equal to $v^e = (1-\alpha)/(n_0\,\rho)$ and must not be greater than the cost of a patent $v = \alpha\, a/(\theta\, n_0\, L)$. Therefore, there is no innovation (and no growth) in this model as long as:

(21) $$\theta \leq \frac{\alpha\, a\rho}{(1-\alpha)\, L} \equiv \theta_1$$

that is as long as the diffusion coefficient is too low relative to the productivity of the r&D process, the subjective discount rate, the elasticity of substitution or the labor supply. Given θ, this case may appear in an economy relatively poor in labor with respect to the R&D productivity, where agents are "impatient" (high ρ) and have a "little" preference for diversity (α near one).

Solutions with sustained growth

In this case, the total number of varieties increases at the exponential rate $g > 0$:

(22) $$n\,(t) = n_0\, e^{gt};$$

The value of a monopoly is decreasing with the number of monopolies and given by

(23) $$v^e\,(t) = \frac{1-\alpha}{(\rho+g)\, n_0\, e^{gt}}$$

and must be equal to the cost of a patent $v = e^{-gt}\, aw\,(t)/(n_0\,\theta)$. This determines a constant wage rate

$$w = \frac{(1-\alpha)\,\theta}{a\,(\rho+g)}$$

and, using (20)

$$g = \frac{\dot{n}}{n} = \frac{\theta}{a}\left[L - \frac{\alpha}{w}\right],$$

allows us to establish a first proposition.

> PROPOSITION 1 : Under infinitely lived patents, if $\theta > \theta_1$, there exists an endogenous balanced growth path [7] with a constant and positive rate
>
> (24) $$g^*_\infty = (1-\alpha)\,\theta\,\frac{L}{a} - \alpha\rho.$$
>
> If $\theta \leq \theta_1$, there is no innovation and no growth.

The rate g^*_∞ depends positively on the labor endowment of the economy, on R&D productivity, on the preference for diversity and on the diffusion coefficient. It depends negatively on the agents' impatience. The degree of

7. Note that g is only the growth rate of the number of varieties. As mentioned in section 2, aggregate growth is measured in terms of utility and is just a positive transformation of g. Indeed, it comes from (2) that $D = n^{1/\alpha}\, L_\alpha$ and that in a steady state, $\dot{D}/D = (1/\alpha)\,\dot{n}/n = g/\alpha$.

non excludability of knowledge, as measured by θ, must be greater than the threshold value θ_1 in order for innovation to be possible. If $\theta < \theta_1$, the spillover effect associated with the R&D process is not high enough to allow for sustained growth.

We turn now to the case of finite patents lifetime.

3.2. Finite Patents Lifetime

When patents lifetime is finite, the dynamics of the model are defined by equations (16)-(19).

Solutions without growth

When $\dot{n} = 0$ on a steady state, there is no patented variety: $\Delta_n = 0$ and $k_n = n_c = n_0$. The cost of a patent is equal to $v = a\tilde{w}/n_0 = a/(n_0 L)$ and is greater than the value of a monopoly:

$$v^e = \frac{(1-\alpha)}{\rho\alpha^{1-\sigma}\, n_0}\,(1 - e^{-\rho T}). \tag{25}$$

The equilibrium condition, $v^e \leq v$ can then be written:

$$1 - e^{-\rho T} \leq \frac{\rho\, a}{L\,(1-\alpha)\,\alpha^{\sigma-1}}. \tag{26}$$

Note that the equilibrium condition does not depend on the diffusion coefficient θ contrary to the condition (21) used in the case of infinitely lived patents. The reason for this, in these last case, is the fact that even when $\dot{n} = 0$, there always exist monopolies. Thus a share of total existing knowledge remains private. When patents lifetime is finite, a steady state without growth is characterized by the absence of any patented variety. Therefore, the whole existing knowledge is public and the particular value of θ is not relevant.

When $\rho \geq (1-\alpha)\,\alpha^{\sigma-1}\,L/a$, condition (26) is satisfied for all T. In this case, there exists an equilibrium path without innovation for any patents lifetime. As before, there is no growth when agents are too impatient, have too low a preference for diversity [8] or when the labor endowment relative to the R&D productivity parameter is too low.

On the contrary, under the following hypothesis

H1. $\rho < (1-\alpha)\,\alpha^{\sigma-1}\,L/a$

there exists an equilibrium path without growth only if patents lifetime is short enough. Indeed, in this case, condition (26) is equivalent to $T \leq T_m$ where T_m is defined by:

$$e^{-\rho T_m} = 1 - \frac{\rho\, a}{L\,(1-\alpha)\,\alpha^{\sigma-1}}; \tag{27}$$

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8. This comes from the fact that $(1-\alpha)\,\alpha^{\sigma-1}$ is a decreasing function of α on [0, 1].

The patents lifetime determines the length of the period during which a monopoly will be able to repay its R&D cost. The shorter is T, the lower is the net rentability of investment. When $T \leq T_m$, investment is never profitable, while it would be so if T were greater than T_m. It is interesting to note that this threshold value increases when L/a decreases or when α increases.

The net effect of ρ on T_m is more difficult to determine. Indeed, it comes from the examination of (25) that ρ has two contrary effects on v^e. On the one hand, a higher ρ-value means that agents are more impatient and are making a lower expectation about the value of a monopoly. Thus, on an equilibrium path without growth, this would allow for higher T_m-values. But on the other hand, ρ measures the weight of the patents lifetime in the valuation of a monopoly. For any given T, the higher is ρ the higher is $(1 - e^{-\rho T})$. Under these conditions, on an equilibrium path without growth, higher ρ-values are consistent with lower T_m-values. Nevertheless, it can be shown that the former effect always dominates [9] and that T_m increases with ρ.

Solutions with sustained growth

We turn now to the most interesting case of our analysis. We determine here the existence condition for a steady state with growth and we analysis how the associated growth rate is affected by the patents lifetime.

On a sustained growth path, the number of varieties is given by (22). The number of competitive varieties is equal to $n_c(t) = n(t-T) = n(t)\, e^{-gT}$. We can then compute the value of a monopoly

$$v^e(t) = \frac{(1-\alpha)\left[1 - e^{-(\rho+g)T}\right]}{n_0\left[1 + (\alpha^{1-\sigma} - 1)\, e^{-gT}\right](\rho+g)}\, e^{-gt} \tag{28}$$

which is decreasing at rate g. Given that the value of a monopoly v^e must be equal to the cost of a patent v, equation (12) allows us to write down the equilibrium value of the wage rate, which is a constant.

$$w = \frac{(1-\alpha)\left[1 - e^{-(\rho+g)T}\right]\left[\theta + (1-\theta)\, e^{-gT}\right]}{a\left[1 + (\alpha^{1-\sigma} - 1)\, e^{-gT}\right](\rho+g)} \tag{29}$$

The law of motion of n, defined by equation (16) is then equivalent to

$$L\left[\theta + (1-\theta)\, e^{-gT}\right] - ag = \frac{\alpha\, a\,(\rho+g)\left[1 + (\alpha^{-\sigma} - 1)\, e^{-gT}\right]}{(1-\alpha)\left[1 - e^{-(\rho+g)\,T}\right]};$$

---

9. By differentiating (27) with respect to T_m and ρ, one gets

$$\frac{dT_m}{d\rho} = \frac{a}{\rho\, L\,(1-\alpha)\,\alpha^{\sigma-1}}\, e^{\rho T_m} - \frac{T_m}{\rho} = A(T_m) - B(T_m).$$

Since $A(0) > B(0)$ and since, under H1, $A'(T_m) > B'(T_m)$, one has $A(T_m) > B(T_m)\, \forall T_m > 0$.

and can be rewritten

$$F(g, T) \equiv A(g, T) - B(g, T) = 0 \tag{30}$$

with:

$$A(g, T) = \frac{1-\alpha}{\alpha a} \frac{1 - e^{-(\rho+g)T}}{\rho + g}$$
$$\text{and} \quad B(g, T) = \frac{1 + (\alpha^{-\sigma} - 1) e^{-gT}}{L(\theta + (1-\theta) e^{-gT}) - ag}.$$

We are looking for solutions to (30) where the growth rate is defined as a function of patents lifetime $g = g(T)$. Any solution $g(T) > 0$ satisfies the equilibrium condition $w > \tilde{w}$.

Unfortunately, it is not possible to write down explicitly such solutions. Nevertheless, assuming that assumption H1 is satisfied – *i.e.* that agents are relatively patient – we define

$$\overline{\theta} = \frac{1 - \rho(\alpha^{1-\sigma} - \alpha) a/L}{\alpha^{1-\sigma} + 1 - \alpha}.$$

Under H1, $\theta_1 < \alpha^\sigma < \overline{\theta}$, and since $\sigma = \sigma(\alpha)$ with

$$\lim_{\alpha \to 0} (\alpha^{1-\sigma} - \alpha) = 1$$

we also have $\overline{\theta} < 1$. Thus, we can establish the following results whose formal proof is given in appendix A.

PROPOSITION 2 : [Existence]
If $T > T_m$, there exists a solution $g^*(T) > 0$ which is unique if $\theta \leq \alpha^\sigma$.

PROPOSITION 3 : [Limit of the solution when $T \to \infty$]
Any solution $g^*(T)$ satisfies:

$$\text{if } \theta \leq \theta_1, \quad \lim_{T \to \infty} g^*(T) = 0$$
$$\text{if } \theta > \theta_1, \quad \lim_{T \to \infty} g^*(T) = g^*_\infty$$

PROPOSITION 4 : [Solutions greater than g^*_∞]
For $\theta_1 < \theta < \overline{\theta}$, equation $F(g, T) = 0$ admits a unique solution $g^*(T) > g^*_\infty$ for large enough T-values.
For $\theta \geq \overline{\theta}$, and for any T, $F(g, T)$ admits no solution $g^*(T) > g^*_\infty$.

It comes from the three previous propositions that:

THEOREM 1. [Maximum growth rate]
T1.1. The equilibrium growth rate is maximized by a finite patents lifetime $\hat{T}$ when $\theta < \overline{\theta}$ and by infinitely lived patents when $\theta \geq \overline{\theta}$.
The finite lifetime $\hat{T}$ and the corresponding maximized growth rate $\hat{g}$ are characterized by the system [10]: $F(\hat{g}, \hat{T}) = 0$ and $F'_T(\hat{g}, \hat{T}) = 0$.
T1.2. When the diffusion coefficient θ tends to $\overline{\theta}$, the increasing function $\hat{g}(\theta)$ tends to the limit $g^*_\infty|_{\theta=\overline{\theta}}$ and $\hat{T}$ tends to infinity.

Part 1 of theorem 1 means that when the diffusion coefficient is sufficiently high, the rentability of R&D investments is maximized when innovators are protected forever. If we compare a situation with infinitely lived patents to a situation with finite patents lifetime, the cost of a patent is higher in the former (under infinitely lived patents, a share $1 - \theta$ of knowledge always remains public) but the market value of a monopoly is also higher. The negative effect due to a restriction in the diffusion of knowledge is more than compensated by the length of the commercial exploitation period.

This is no more the case when the diffusion coefficient becomes lower than $\overline{\theta}$. The growth maximizing patents lifetime is then finite. In this case, an economy characterized by $T < \hat{T}$ would be growing slower because the decrease of the commercial exploitation period is not compensated by a sufficient decrease of the cost of a patent. Conversely, an economy with $T > \hat{T}$ would be also growing slower because the increase of the commercial exploitation period is more than compensated by an increase of the cost of a patent.

The diffusion coefficient is always growth promoting when $T = \hat{T}$. Part 2 of theorem 1 show that $\hat{g}$ is an increasing function of θ when $\theta < \overline{\theta}$ and equation (24) show that g^*_∞ is also an increasing function of θ. An increase in θ increases the knowledge spillover effect, and therefore decreases the cost of a patent.

4 Welfare Analysis

Now we have identified the growth-maximizing patents lifetime, we examine what could be the optimal one. By "optimal", we mean the duration that maximizes the utility function (1) taken as a social welfare index. It must be noted that manipulating the patents lifetime allows one only to reach a second-best equilibrium. Indeed, the patent system (when T is finite) induces a price distortion on the differentiated goods market. Furthermore, introducing a new variety induces three different

10. The uniqueness of the pair $(\hat{g}, \hat{T})$ is proved in appendix A.

external effects: a rise in the level of public knowledge, a decrease in the instantaneous profit earned by a monopolist and an increase of households' utility. These effects make impossible the coincidence of the competitive equilibrium with the first-best one.

4.1. **Optimal Growth Rate**

The first-best equilibrium of this model corresponds to a situation where the saving and investment decisions are taken by a social planner. In this case, the patents do not exist and the problem of knowledge diffusion disappears. Indeed, the social planner use the whole knowledge developed through R&D activities and $k_n = n$. The problem to be solved is then very simple and consists of maximizing U with respect to the resource constraint:

$$L = a\,\frac{\dot{n}}{n} + \int_0^n x\,(i)\,di.$$

Since (i) the static utility index treats all the varieties symmetrically, (ii) the marginal utility associated with the consumption of each good is decreasing $(\alpha < 1)$, and (iii) the production functions are identical, it follows that the optimal static allocation of labor among branches is symmetric. We then have $x\,(i) = x$ and the first-best equilibrium is the solution of the program

$$(31) \qquad \begin{aligned} &\max_{[x\,(t)]_0^\infty} U = \int_0^\infty e^{-\rho t}\left[\frac{1}{\alpha}\ln n\,(t) + \ln x\,(t)\right] dt \\ &\text{s.t. } a\,\frac{\dot{n}\,(t)}{n\,(t)} = L - n\,(t)\,x\,(t) \qquad \text{and} \qquad n\,(t) \geq 0\,\forall\, t \geq 0. \end{aligned}$$

The solution corresponds to the growth rate

$$(32) \qquad g^0 = \frac{1}{1-\alpha}\left[(1-\alpha)\,\frac{L}{a} - \alpha\rho\right] = \frac{g^*_\infty|_{\theta=1}}{1-\alpha}.$$

The socially optimal growth rate g^0 is thus always greater than $g^*_\infty|_{\theta=1}$ which is the highest feasible growth rate in the competitive economy. The latter is thus always characterized by underinvestment with respect to the Pareto-optimal situation.

With this result, one could imagine that every growth increasing factor is also welfare improving. In this case, the optimal patents lifetime would be the growth maximizing one. In a second-best analysis, nonever this is the case, as we show in the next paragraph.

4.2. **Optimal Patents Lifetime**

In order to determine the optimal patents lifetime, we need to compute the value of the utility function U on a sustained growth path. The representative household's utility is given by

$$U = \int_0^\infty e^{-\rho t} \ln D\,(t) dt$$

with $n(t) = n_0 e^{gt}$. The static utility index (2) is

$$D = (n_c x_c^\alpha + \Delta_n x_m^\alpha)^{1/\alpha}.$$

where x_c and x_m are given by (14). Thus we have

$$D(t) = \frac{\alpha}{w}\left[1 + (\alpha^{1-\sigma} - 1)\, e^{-gT}\right]^{\frac{1-\alpha}{\alpha}} (n_0 e^{gt})^{\frac{1-\alpha}{\alpha}}$$

which becomes, using the expression (29) of w

$$D(t) = N(g, T)\,(e^{gt})^{\frac{1-\alpha}{\alpha}}$$

with

$$N(g, T) = \frac{\alpha\, n_0^{(1/\alpha)-1}\, a\,[\rho + g]\,[1 + (\alpha^{1-\sigma} - 1)\, e^{-gT}]^{1/\alpha}}{(1-\alpha)\,[1 - e^{-(\rho+g)T}]\,[\theta + (1-\theta)\, e^{-gT}]}.$$

We then deduce the value of intertemporal utility on the equilibrium path characterized by $g = g^*(T)$.

(33) $$U^*(T) \equiv U[g^*(T), T] = \frac{1}{\rho}\ln N[g^*(T), T] + \frac{1-\alpha}{\alpha}\,\frac{g^*(T)}{\rho^2}$$

Defining $N^*(T) \equiv N[g^*(T), T]$, the derivative of U^* with respect to the patents lifetime is therefore defined by

(34) $$\frac{dU^*(T)}{dT} = \frac{1}{\rho\, N^*(T)}\,\frac{dN^*(T)}{dT} + \frac{1}{\rho^2}\,\frac{1-\alpha}{\alpha}\,\frac{dg^*(T)}{dT}.$$

and its sign is studied in appendix B where we consider a third threshold value for the diffusion coefficient:

(35) $$\theta_3 = \frac{\alpha}{\alpha^{1-\sigma} - 1 + \alpha}.$$

We now state the following hypothesis:

H.2. $\rho < \dfrac{(1-\alpha)(\alpha^{1-\sigma} - 1 - \alpha)}{(\alpha^{1-\sigma} - \alpha)(\alpha^{1-\sigma} - 1 + \alpha)}\,\dfrac{L}{a}.$

This hypothesis is more restrictive than H1 – *i.e.*, it imposes a lower value for the subjective discount rate ρ.

THEOREM 2 : [Optimal patents lifetime]
If the subjective discount rate satisfies H2, we have $\theta_3 < \overline{\theta}$ and for all $\theta \in [\theta_3, \overline{\theta}]$

$$\frac{dg^*(T)}{dT} \leq 0 \Rightarrow \frac{dU^*(T)}{dT} < 0.$$

Thus, the optimal patents lifetime – *i.e.* the patents lifetime that maximizes the representative household's utility – is finite and shorter than the value $\hat{T}$ that maximizes the growth rate.

Unfortunately, we have not been able to conduct a complete algebraic analysis for θ-values belonging to $[0, \overline{\theta}]$ and so we have proceeded with numerical simulations. These simulations, detailed in sub-section 4.3, show that under H2, theorem 2 holds for the whole interval $[0, \overline{\theta}]$.

This theorem means that when the instantaneous diffusion of knowledge is relatively low, the second best patents lifetime is shorter than the growth maximizing one. Although private investment is always too low to be optimal in a first best analysis, we see here that in a second best framework, the resulting effect of the different externalities may induce overinvestment. When the only policy tool is the patents lifetime, the arbitrage between industrial profitability and welfare then entails the choice of legal commercial protection that reduces the incentives for investment faced by private innovators.

4.3. Numerical Analysis

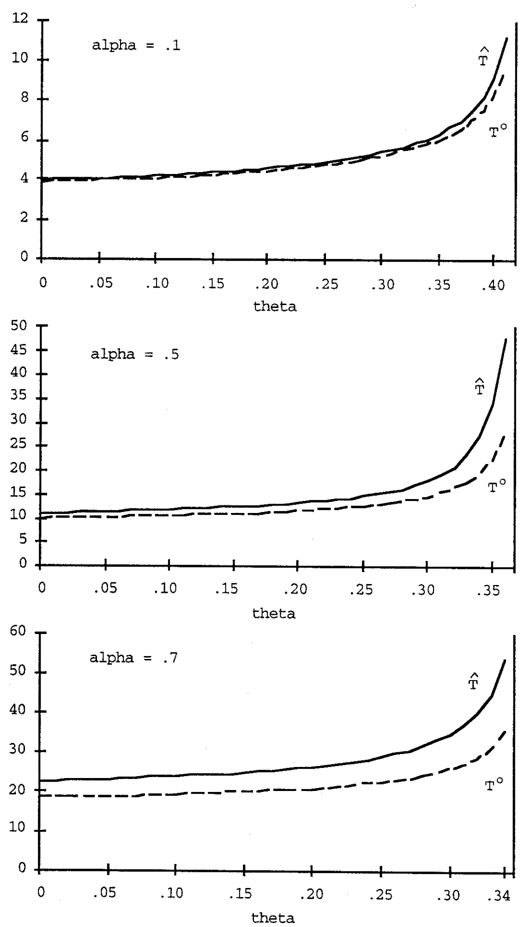
The aim of our numerical analysis is to obtain two lists of T-values, given as function of θ, corresponding respectively to the growth maximizing and to the optimal patents lifetime on the interval $[0, \overline{\theta}]$. We restrict the analysis to this interval since it corresponds to the case where growth is maximized by a *finite* patents lifetime. Using theorem 1, we compute $\hat{T}(\theta)$ as the solution of the system $\{F(\hat{g}, \hat{T}) = 0,\ F'_T(\hat{g}, \hat{T}) = 0\}$, and we repeat this for θ starting at 0 and going to $\overline{\theta}$ by steps of .01. This gives us the plain curve corresponding to the collection of points $\{\theta, \hat{T}(\theta)\}$ that appears in figure 1. We use the same procedure in order to determine the list of T^0-values that maximize social welfare for $\theta \in [0, \overline{\theta}]$. Here, $T^0(\theta)$ is computed as the solution to the system $\{F(g^0, T^0) = 0,\ U'_T(g^0, T^0) = 0\}$.

The following values have been assigned to the parameters of the model. We have chosen to normalize to one the labor supply L and the productivity parameter a. These two parameters reflect the resource endowment of the economy and play a role mainly on the level of the growth rate. We have fixed to one the value of the initial variety level $n_0 \equiv n(0)$ in order to have always $n(t) = e^{gt}$. The values of ρ and α have been fixed in such a way that assumption H2 – and therefore assumption H1 – is satisfied. More precisely, we have assigned a usual value to the subjective discount rate, which is also the interest rate here, assuming $\rho = .05 = 5\%$. Then, H2 is satisfied for $0 \leq \alpha \leq .7$. This last parameter, that indicates the degree of substitutability between two varieties, is actually the only one (with θ) whose modification could entail qualitative modifications of our results. We have conducted the numerical analysis, described above, for three α-values, namely .1, .5 and .7. In each of these three cases – that correspond to the three graphics in figure 1 –, the main result of theorem 2 is satisfied. Then, as the other parameters, α has only a "level effect" on the equilibrium values of g, U and on the T-values that maximize either g or U. Having tried many different combinations, we can say more generally that the model is qualitatively robust to changes in the parameter values as long as these satisfy assumptions H1 and H2.

Examination of figure 1 confirms the results of theorem 1. The patents lifetime that maximizes the growth rate is finite as long as the diffusion parameter remains lower than $\overline{\theta}$. Furthermore, $\hat{T}$ is an increasing function

FIGURE 1

Numerical simulations for $\theta < \overline{\theta}$. Results of theorem 2 hold when H2 is satisfied.



of θ that tends to infinity when θ tends to $\overline{\theta}$. The numerical analysis also extends the result of theorem 2 to a set of θ-values belonging to $[0, \theta_3]$. By theorem 2, we know that the optimal patents lifetime T^0 is lower than the growth maximizing one when $\theta \in [\theta_3, \overline{\theta}]$. Simulations show that this is still the case when $\theta < \theta_3$.

We observe that the collection of points $\{\theta, T^0(\theta)\}$ qualitatively behaves like the collection $\{\theta, \hat{T}(\theta)\}$, that is, $T^0(\theta)$ is increasing with θ. This shows that, if one interprets T as a means for the government to increase or reduce the effective excludability of knowledge, the lower the "natural" degree of non-excludability θ, the higher the "artificial" one T. In other words, the government, in order to reach a second best equilibrium, tends to compensate for the lack of knowledge diffusion by imposing a finite patents lifetime.

Finally, one can see that the lower is α, the smaller the difference between T^0 and $\hat{T}$ and the lower the absolute value of T^0 and $\hat{T}$. This can be explained by the fact that when α tends to 0, the perception of diversity becomes maximal, and therefore so does the profit opportunity for innovators. Thus, everything being equal elsewhere, a decrease in α makes the maximum growth rate consistent with a shorter patents lifetime because a shorter T induces a lower R&D cost. For the same kind of reasons, the optimal patents lifetime can also be reduced. The reduction of the gap between T^0 and $\hat{T}$ when α tends to zero is more difficult to explain but comes from the fact that the weight of the rate of product variety expansion in the utility function becomes higher with respect to the weight of price distortions due to monopolies. These properties are summarized in the following example (table 1) which gives the value of the optimal and growth maximizing patents lifetime when only one third of the knowledge becomes instantaneously public.

TABLE 1

Comparison of the growth maximizing and the optimal patents lifetime when the diffusion parameter is equal to 1/3.

$\theta = 1/3$	$\hat{T}$	T^0
$\alpha = .1$	5.9	5.7 (-4%)
$\alpha = .5$	23.4	17.6 (-24%)
$\alpha = .7$	45.1	31.43 (-30%)

5 Conclusion

We have constructed an innovation based endogenous growth model based on GROSSMAN and HELPMAN [1991a, ch. 3) and on JUDD [1985]. We depart from these two models by assuming that patents are not only a commercial protection for innovators – the only function of a patent in Grossman and Helpman – but also allow innovators to keep private a share of technological information as long as the patent protection lasts – Judd assumes a full property right so that endogenous growth is not possible in his model.

The fact that a patent allows an innovator to keep secrecy on a part of the knowledge he has discovered reduces the knowledge spillover effect that makes growth sustainable. Therefore, the design of the patents policy must take into account not only the usual arbitrage between private profitability and distortions due to the existence of monopolies, but also the diffusion of knowledge that directly influences the pace of growth.

In this framework, we have provided a detailed analysis of the equilibrium growth path. We have shown that growth is maximized by a finite patents lifetime when the property right concerns a large share of knowledge. At the contrary, growth is maximized by infinitely lived patents when this share is low. The same kind of reasoning applies for the optimal patents lifetime. In particular, we show that when growth is maximized by finite patents lifetime, the optimal lifetime is also finite and shorter than the growth maximizing one. The reason for this is quite intuitive. When the "natural" diffusion of knowledge is low, the government can compensate the lack of public knowledge by choosing a lifetime shorter than the one that would have maximized innovators' profitability. Manipulating patents lifetime may then become, in a second best analysis, a means to modify the effective excludability of knowledge.

In this paper, we have treated θ as a parameter. One possible extension would be to consider an endogenous value for the share of public knowledge. This would be more appropriate to the intentional secrecy interpretation of $\theta < 1$. But such a hypothesis would require a very different treatment of the commercial exploitation of an innovation. Here, a patent always ensures full commercial protection; therefore individual innovators have no incentives for manipulating the value of θ. Endogenous determination of firms secrecy policy makes sense only if reducing θ has a direct positive effect on the profitability of an innovation. One way to model this idea may be to take into account the very high costs induced by patent protection and to distinguish innovations according to their degree of knowledge appropriability. Then, innovators must choose between patenting or not, and θ can be reinterpreted as the share of patented innovations. Obviously, this framework is quite different from the one adopted here and its exploration is left to further research.

A. Proof of Theorem 1

We study here the solution $g^*(T)$ which is implicitly defined by equation (30)

$$F(g,T) \equiv A(g,T) - B(g,T) = 0$$

with:

$$A(g,T) = \frac{1-\alpha}{\alpha a}\,\frac{1-e^{-(\rho+g)T}}{\rho+g}$$
$$\text{and} \quad B(g,T) = \frac{1+(\alpha^{-\sigma}-1)\,e^{-gT}}{L(\theta+(1-\theta)\,e^{-gT})-ag}.$$

First, we proceed with some preliminary results.

I. Any solution g satisfies $M(g,T) \equiv L(\theta+(1-\theta)\,e^{-gT})-ag > 0$. For a given T, $M(g,T)$ is decreasing and positive on the interval $[0,\tilde{g}(T)]$ where $\tilde{g}(T)$, defined by $M[\tilde{g}(T),T] = 0$, is decreasing with T.

II. The sign of the derivative of $A(g,T)$ with respect to g can be deduced from the following computation. For any u, $u > 0$, the derivative of the function $(1-e^{-u})/u$ is equal to $u^{-2}\,h(u)$ with $h(u) = ue^{-u} - 1 + e^{-u}$. Since $h'(u) = -ue^{-u} < 0$ for $u > 0$, we have $h(u) < h(0) = 0$. It follows that for any $g \geq 0$ and for a given T, the function $A(g,T)$ is decreasing in g.

III. We now define the sign of the derivative of $B(g,T)$ with respect to g in order to get the sign of $F'_g(g,T)$.

$$B'_g(g,T) = \frac{Te^{-gT}\,m(g) + a\,[1+(\alpha^{-\sigma}-1)\,e^{-gT}]}{M(g,T)^2}$$

where

$$m(g) \equiv L\,(1-\alpha^{-\sigma}\,\theta) + (\alpha^{-\sigma}-1)\,ag.$$

Note that $m(g)$ is an increasing function of g.

A.1. Existence. When $\theta \leq \alpha^{\sigma}$, $m(g) > 0$, $B'_g > 0$, $F'_g < 0$, and there exists at most one solution.

In the general case, we have

$$F(0,T) = \frac{(1-\alpha)\,(1-e^{-\rho T})}{\alpha\,a\,\rho} - \frac{\alpha^{-\sigma}}{L}$$

$F(0,T) > 0$ iff $T > T_m$ and $\lim_{g\to\tilde{g}(T)} F(g,T) = -\infty$. Then, if $T > T_m$, $F(g,T) = 0$ admits a solution. This proves proposition 1.

A.2. Limits of the solution when $T \to \infty$. Consider a sequence $T_n \to \infty$ such that $\lim_{n\to\infty} g^*(T_n) = \overline{g} > 0$. We have then:

$$0 = \lim_{n\to\infty} F(g^*(T_n), T_n) = \frac{1-\alpha}{\alpha a}\frac{1}{\rho+\overline{g}} - \frac{1}{L\theta - a\overline{g}}$$

which implies: $\theta > \theta_1$ and $\overline{g} = g^*_\infty$.

As a consequence, when $\theta \leq \theta_1$, 0 is the unique limit point of $g^*(T)$ and we have:

$$\text{if } \theta \leq \theta_1, \ \lim_{T\to\infty} g^*(T) = 0.$$

This proves proposition 3.

A.3. Solutions greater than g^*_∞. Here, we proceed in two steps.

I. Variations of $F(g, T)$ with respect to T

For given g, $F(g, T)$ is a function of T defined, continuous and differentiable on $[0, \overline{T}(g)[= \{T > 0; \tilde{g}(T) > g\}$ and we have:

$$\begin{aligned} \overline{T}(g) &= +\infty \qquad \text{if} \quad g \leq L\theta/a \\ \overline{T}(g) &< +\infty \qquad \text{if} \quad L\theta/a < g < L/a. \end{aligned}$$

The derivative of F with respect to T verifies:

$$\begin{aligned} e^{gT} F'_T(g, T) &= \frac{1-\alpha}{\alpha a} e^{-\rho T} - \frac{g m(g)}{M(g, T)^2} \equiv \Psi(g, T) \\ \Psi'_T(g, T) &= -\frac{1-\alpha}{\alpha a}\rho e^{-\rho T} - \frac{g^2 m(g)(1-\theta)e^{-gT}}{M(g, T)^2} \end{aligned}$$

• When $m(g) \leq 0$, we have $F'_T(g, T) > 0$ and F is an increasing function of T within $[0, \overline{T}(g)[$.

• When $m(g) > 0$, we have $\Psi'_T(g, T) < 0$ and $\lim_{T\to\overline{T}} \Psi(g, T) < 0$.

– If $\Psi(g, T) \leq 0$, F is an increasing function of T within $[0, \overline{T}(g)[$.

– If $\Psi(g, T) > 0$, F is first increasing, reaches a maximum at $T = T_0$, and then becomes decreasing.

II. Comparison of $g^*(T)$ and g^*_∞.

We assume $\theta > \theta_1$. We have $\overline{m} \equiv m(g^*_\infty) > 0$ iff

$$\theta < \overline{q} = \frac{1 - \rho(\alpha^{1-\sigma} - \alpha)a/L}{\alpha^{1-\sigma} + 1 - \alpha}.$$

If $\overline{m} > 0$, there exists at most one solution $g^*(T) > g^*_\infty$ because $F'_g(g, T) < 0$ for $g > g^*_\infty$. From **I.**, we have $F'_T(g^*_\infty, T) < 0$ for large enough T and $\lim_{T\to\infty} F(g^*_\infty, T) = 0$. We then deduce that $F(g^*_\infty, T)$ takes positive values and that one unique solution $g^*(T) > g^*_\infty$ necessary exists.

If $\overline{m} \leq 0$, then for all g such that $m(g) \leq 0$, we have $F'_T(g, T) > 0$ and

$$F(g, +\infty) = \frac{g^*_\infty - g}{\alpha(\rho+g)(L\theta - ag)} \leq 0 \quad \text{if } g \geq g^*_\infty.$$

This implies $F(g, T) < 0$ for all T and for all $g \geq g^*_\infty$ such that $m(g) \leq 0$. Let us choose $g_0 \geq g^*_\infty$ such that $m(g_0) = 0$. For $g > g_0$, we have $F'_g(g, T) < 0$ and then

$$F(g, T) < F(g_0, T) < 0.$$

In this case, there does not exist a solution $g^*(T) \geq g^*_\infty$ with a finite T-value. This proves proposition 4.

Proof of Theorem 1.

We are now studying the shape of the implicit function $g^*(T)$. Consider $b \geq 0$ such that $m(b) \geq 0$ and such that there exists only one solution $g^*(T) > b$ to equation $F(g, T) = 0$ for $T \in (T_1, T_2)$. When T belongs to this interval, we have $F'_g[g^*(T), T] < 0$ since $g^*(T) > b \Rightarrow m[g^*(T)] > b > 0 \Rightarrow B'_g[g^*(T), T] > 0$. From the implicit function theorem, it follows that $g^*(T)$ is differentiable and that

$$\frac{dg^*(T)}{dT} = -\frac{F'_T[g^*(T), T]}{F'_g[g^*(T), T]}$$

which has the same sign that $F'_T[g^*(T), T]$.

Lemma 1. If there exists a $\hat{T}$ such that $F'_T[g^*(\hat{T}), \hat{T}] = 0$, then for all $T \in]\hat{T}, T_2[$, we have:

$$F'_T[g^*(T), T] < 0 \qquad \text{and} \qquad \frac{dg^*(T)}{dT} < 0.$$

Proof. We define $X^*(T) = e^{g^*(T)T} F'_T[g^*(T), T]$:

$$X^*(T) = \frac{1-\alpha}{\alpha a} e^{-\rho T} - f[g^*(T), T] \text{ where } f(g, T) = \frac{gm(g)}{M(g, T)^2}$$

Since $M(g, T)$ is decreasing in g and T, $f(g, T)$ verifies $f'_g > 0$ and $f'_T > 0$.

$$\frac{dX^*(T)}{dT} = -\frac{1-\alpha}{\alpha a}\rho e^{-\rho T} - f'_T[g^*(T), T] - f'_g[g^*(T), T]\frac{dg^*(T)}{dT}$$

Then, we have

$$\frac{dX^*(T)}{dT} = -S_1(T) - S_2(T)X^*(T)$$

with

$$S_1(T) = \frac{1-\alpha}{\alpha a}\rho e^{-\rho T} + f'_T[g^*(T), T] > 0$$

$$S_2(T) = \frac{f'_g[g^*(T), T]e^{-g^*(T)T}}{-F'_g[g^*(T), T]} > 0.$$

Near $T = \hat{T}$, $dX^*(T)/dT < 0$. Since for $T = \hat{T}$, $X^*(\hat{T}) = 0$, we have $X^*(T) < 0$ for $T \in]\hat{T}, T_2[$ and $F'_T[g^*(T), T] < 0$, implying:

$$\frac{dg^*(T)}{dT} = -\frac{F'_T[g^*(T), T]}{F'_g[g^*(T), T]} < 0. \quad \square$$

Proof of Theorem 1. [Continuation]

Lemma 1 can be used in the case $\theta < \overline{\theta}$ where there exists a unique solution $g^*(T) > 0$ for $T \in]T_m, +\infty[$ if $\theta \leq \alpha^\sigma$ or $g^*(T) > g^*_\infty$ for $T \in]T_1, +\infty[$ if $\alpha^\sigma < \theta < \overline{q}$. The differentiable function $g^*(T)$ reaches its maximum at a point $\hat{T} \in]T_m, +\infty[$ in the first case and at a point $\hat{T} \in]T_1, +\infty[$ in the second, such that $F'_T[g^*(\hat{T}), \hat{T}] = 0$. Thanks to the lemma, we know that no other local extremum exists in $T_l \neq \hat{T}$. By lemma 1, it cannot exist $T_l > \hat{T}$. If it did exist $T_l < \hat{T}$, we should have $F'_T[g^*(T_l), T_l] = 0$ and $dg^*(T)/dT < 0$ for all $T > T_l$.

When $\theta < \overline{\theta}$, the growth rate reaches a maximum $\hat{g} \equiv g^*(\hat{T})$ at $T = \hat{T}$. Let us now study the variations of $\hat{T}$ and $\hat{g}$ with respect to θ. By differentiating $F(\hat{g}, \hat{T}) = 0$, we get:

$$F'_g(\hat{g}, \hat{T})\, d\hat{g} + F'_T(\hat{g}, \hat{T})\, d\hat{T} + [1 + (\alpha^{-\sigma} - 1)\, e^{-\hat{g}\hat{T}}]\, \frac{(1 - e^{-\hat{g}\hat{T}})\, L}{M(\hat{g}, \hat{T})^2}\, d\theta = 0.$$

Since $F'_g(\hat{g}, \hat{T}) < 0$ and $F'_T(\hat{g}, \hat{T}) = 0$, we deduce:

$$\frac{d\hat{g}}{d\theta} > 0.$$

We have $\lim_{\theta \to \overline{\theta}} \hat{g}(\theta) = g^*_\infty|_{\theta = \overline{\theta}}$. At this point:

$$m(g^*_\infty|_{\theta = \overline{\theta}}) = 0.$$

Then:

$$\lim_{\theta \to \overline{\theta},\, \theta < \overline{\theta}} \hat{T}(\theta) = +\infty.$$

The previous results are synthesized in figure 2 and complete the proof of theorem 1.

FIGURE 2

Summary of the results (appendix A).

0 — θ_1 — α^σ — $\overline{\theta}$

<table>
<tr><td colspan="2">Unique $g^*(T) > 0$ for $T > T_m$</td><td colspan="2">$\lim_{T \to \infty} g^*(T) = g^*_\infty$</td></tr>
<tr><td>$\lim_{T \to \infty} g^*(T) = 0$</td><td colspan="2">Unique $g^*(T) > g^*_\infty$ for $T > T_1$</td><td>No $g^*(T) > g^*_\infty$</td></tr>
<tr><td colspan="3">$\frac{d\hat{g}}{d\theta} > 0,\ \lim_{\theta \to \overline{\theta}} \hat{g} = g^*_\infty|_{\theta = \hat{\theta}}$ and $\lim_{\theta \to \overline{\theta},\, \theta < \overline{\theta}} \hat{T} = +\infty$</td><td rowspan="2">Growth is maximized with infinitely lived patents.</td></tr>
<tr><td colspan="3">It always exists a positive growth rate
for large enough T.
Growth is maximized by a finite patents lifetime.</td></tr>
</table>

B. Proof of Theorem 2

Our goal here is to study the sign of the derivative defined by (34).

$$\frac{dU^*}{dT} = \frac{1}{\rho N\left[g^*(T),\,T\right]}\,\frac{d}{dT}\,N\left[g^*(T),\,T\right] + \frac{1}{\rho^2}\left(\frac{1}{\alpha}-1\right)\frac{dg^*(T)}{dt}.$$

We define $Y^*(T) = e^{-g^*(T)T}$. Then:

(36)

$$N\left[g^*(T),\,T\right] = \frac{\alpha\, n_0^{(1/\alpha)-1}\, a\left[\rho+g^*(T)\right]\left[1+\left(\alpha^{1-\sigma}-1\right)Y^*(T)\right]^{1/\alpha}}{(1-\alpha)\left[1-Y^*(T)\,e^{-\rho T}\right]\left[\theta+(1-\theta)\,Y^*(T)\right]}$$

$$\frac{1}{N\left[g^*(T),\,T\right]}\,\frac{dN\left[g^*(T),\,T\right]}{dT} = -\frac{\rho\, Y^*(T)\, e^{-\rho T}}{1-Y^*(T)\,e^{-\rho T}} + \frac{1}{\rho+g^*(T)}\,\frac{dg^*(T)}{dT} + P^*\,\frac{dY^*(T)}{dT}$$

where

$$P^* = \frac{\left(\alpha^{1-\sigma}-1+\alpha\right)\theta-\alpha+\left(\alpha^{1-\sigma}-1\right)(1-\alpha)(1-\theta)\,Y^*(T)}{\alpha\left[1+\left(\alpha^{1-\sigma}-1\right)Y^*(T)\right]\left[\theta+(1-\theta)\,Y^*(T)\right]} + \frac{e^{-\rho T}}{1-Y^*(T)\,e^{-\rho T}}$$

A sufficient condition for $P^* > 0$ for all T is $\theta \geq \theta_3$ with:

$$\theta_3 = \frac{\alpha}{\alpha^{1-\sigma}-1+\alpha}.$$

This condition is necessary for:

$$\lim_{T\to+\infty} P^* = \frac{\left(\alpha^{1-\sigma}-1\right)\theta-\alpha}{\alpha\theta} \geq 0.$$

We also have:

$$\frac{dY^*(T)}{dT} = -e^{-g^*(T)T}\left(g^*(T)+T\,\frac{dg^*(T)}{dT}\right).$$

Since we have $dg^*(T)/dT = -F_T'/F_g'$,

$$\left(g^*+T\,\frac{dg^*}{dT}\right)F_g' = g^*\,F_g' - T F_T'$$

$$= g^*\,A_g' - \frac{a g^*\left[1+\left(\alpha^{1-\sigma}-1\right)e^{-g^*T}\right]}{M\left(g^*,\,T\right)^2} - T\left(\frac{1-\alpha}{\alpha a}\right)e^{-(\rho+g^*)\,T}.$$

Then, for all T, $\left(g^*+T\,dg^*/dT\right)F_g' < 0$. Since $F_g'\left(g^*,\,T\right) < 0$ when $\theta_1 < \theta < \overline{\theta}$, we deduce that $\left(g^*+T\,dg^*/dT\right) > 0$ and that $dY^*/dT < 0$.

Lemma 2: If $\theta \geq \theta_3$, then:

$$\frac{dg^*(T)}{dT} \leq 0 \Rightarrow \frac{1}{N[g^*(T), T]} \frac{dN[g^*(T), T]}{dT} < 0.$$

Proof: Indeed, in this case we have $P^* \geq 0$ and the three terms on the RHS of (36) are respectively negative, negative or nil if $dg^*/dT \leq 0$, and negative or nil. □

Under H2, we have $\theta_3 < \overline{\theta}$. Then, by lemma 2 and (34), it comes that when θ belongs to $[\theta_3, \overline{\theta}]$, $dg^*/dT \leq 0 \Rightarrow dU^*/dT < 0$, which proves theorem 2. □

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