ANNALES D'ÉCONOMIE ET DE STATISTIQUE. – N° 48 — 1997

# Preference Characterisation and Indirect Allais Coefficients

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**ABSTRACT.** – In this paper, I propose a new way of characterising the consumer's preferences. It is based on the Hessian of the indirect utility function, after this matrix has been purified form its cardinality by means of a procedure suggested by Allais (1943). The properties of this matrix are investigated, and the indirect Allais classification of commodity interactions is compared with more standard classification rules, both theoretically and empirically.

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## Caractérisation des préférences et coefficients d'Allais indirectes

**RÉSUMÉ.** – Dans ce papier, je propose une nouvelle façon de caractériser les préférences du consommateur. Elle est basée sur la hessienne de la fonction d'utilité indirecte, une fois épurée de cette matrice sa cardinalité; suivant une procédure suggérée par Allais (1943). Les propriétés de cette matrice sont étudiées, et la classification des interactions des biens est comparée avec les classifications plus courantes, aussi bien de façon théorique que de façon empirique.

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I am indebted to Anton Barten and Leon Bettendorf for helpful discussions and comments on an earlier version. I should also like to thank an anonymous referee, Kjell Salvanes and seminar participants at the University of Antwerp and at the HCM-meeting *Microeconometrics of public policy issues* at KULeuven (April 1995) for comments. Of course, I bear the sole responsibility for remaining flaws. This paper was started when I was at SESO-UFSIA (University of Antwerp); support there, as well as financial support from EU (through TMR grant no. ERBFMBICT950153) is gratefully acknowledged.

# 1 Introduction

The question whether two commodities are each other's complement or substitute and how this can be inferred from market data, is an old one and most economists will agree by now that its answer should be ordinal, that it should reflect features of the consumer's preference ordering rather than of one particular representation of that ordering. Two classifications which satisfy this criterion are the signs and absolute values of the Slutsky and Antonelli substitution effects-what HICKS (1956, ch 16) called the degree of $p$- and $q$-complementarity, respectively. Both the Slutsky and Antonelli coefficients have the property that they stem from the Hessian matrix of the expenditure and distance function, respectively. A third classification was provided by ALLAIS (1943, p 147). He proposed to look at the elements of the direct utility function's Hessian, after this matrix has been "ordinalised". In a first instant, Allais' idea was overlooked, but its usefulness was stressed by CHARETTE & BRONSARD [1975], BRONSARD & SALVAS-BRONSARD [1988] and BARTEN [1990]. Using the apparatus of modern consumer theory, BARTEN [1990] makes explicit the ordinalisation procedure and shows how the matrix of Allais coefficients-which I shall henceforth call the *direct Allais matrix*-can easily be retrieved from the Rotterdam parameterisation of either the Slutsky and Antonelli matrix.

These three types of classification all have in common that they are based on the (transformed) Hessian of a function which completely describes the preference ordering. In this paper, it is my intention to explore a fourth characterisation of preferences, which closes the circle in a natural way. Indeed, under standard regularity assumptions, also the indirect utility function completely describes the consumer's preference order. Of course, this functions is of a cardinal nature and the elements of its Hessian are not invariant to different cardinalisation of that function. But I shall argue that this cardinality can be removed by exactly the same type of transformation which ALLAIS [1943] applied on the Hessian of the direct utility function. The ensuing matrix-which I shall term the *indirect Allais matrix*-invites for a fourth classification and can be considered of as the dual to the direct Allais matrix [1].

A particular feature of the Slutsky rule is the relative dominance of substitution: with every diagonal element of the Slutsky matrix being negative, the homogeneity condition implies positivity of at least one off-

1. These four classifications are by no means exhaustive. MADDEN (1991), for instance, presents other classifications, indexed on the set of rationed commodities–the $R$-classifications. The idea is that when a consumer faces a quantity constraint w.r.t. some commodities, the compensated price effects on the quantities purchased of the unrationed commodities and the compensated quantity effects on the virtual prices of the rationed commodities may be used to classify the interactions. If the subset of constrainted goods is empty, the $R$-classification coincides with the Hicksian $p$-classification, while the $q$-classification emerges when the quantities of all but one good are fixed.

diagonal element. Thus, under the Slutsky rule, every commodity interacts with at least one other commodity as a substitute. In this paper, I will show that the indirect Allais matrix does not give rise to such dominance while at the same time being positively (that is, not inversely) related to the Slutsky matrix.

The outline of the paper is as follows. In the next section, I shall introduce notation and briefly remind the reader of the Rotterdam parameterisation of regular and inverse demand systems. Indirect Allais coefficients are introduced in section 3. There, I shall stress their ordinal character and argue that these coefficients partially control for changes in the welfare loss a consumer experiences because of a price increase. The relationship of the indirect Allais matrix to the Slutsky, Antonelli and direct Allais matrices is also given. In section 4, I will deliberate on the pros and cons of the different classifications. Section 5 provides an empirical illustration of the four different criteria to classify commodities using a demand system for foodstuffs estimated by SCHOKKAERT & VAN DER WEE [1988] on data of the 16th century Lier Beguinage. Concluding remarks are collected in section 6.

# 2 Rotterdam Parameterisation of Regular and Inverse Demand Systems–a Brief Reminder

I shall assume that preferences are representable by a direct utility function $u(\cdot) : R_+^n \rightarrow R$, which is monotonic, strongly quasi concave and differentiable at least two times. The vector of commodities will be denoted as $q \in R_+^n$ [2]. The consumer's opportunity set is defined by the price vector $p \in R_+^n$ and exogenous income $m$. However, it will prove useful to work with the normalised price vector $\pi =_{\text{def}} \frac{1}{m} p$. The consumer's problem is then to find the commodity bundle which solves the problem $\max_q \{u(q) | \pi' q = 1\}$. For an interior solution–which I shall assume throughout–the optimal bundle $q(\pi)$ must necessarily satisfy the first order condition $u_q = \lambda(\pi)\pi$, where $u_q$ is the vector of marginal utilities evaluated at the bundle $q(\pi)$, and $\lambda(\pi)$ is the Lagrange multiplier to the budget constraint.

Let $w_i$ be the budget share of commodity $i$, i.e. $w_i = \pi_i q_i$, and let $w$ be the vector of budget shares. Using a $\wedge$ above a vector to denote the diagonal matrix with the vector elements on its diagonal, the Rotterdam

2. A vector is always a column vector. Row vectors will be transposed column vectors. The transposed of the column vector $x$ will be denoted as $x'$.

parameterisation of the regular demand system $q(\pi)$ may then be written as follows in differential form:

$$(1) \qquad \hat{w}\,\mathrm{dln}q = b\left[-w'\,\mathrm{dln}\pi\right] + S\,\mathrm{dln}\pi$$

In this system, the vector $b$ is the vector of marginal propensities to spend on the different commodities (since $-w'\,\mathrm{dln}\,\pi = \mathrm{dln}\,m - w'\,\mathrm{dln}\,p$, the logarithmic change in real income) and $S$ is a symmetric matrix whose typical element $s_{ij}$ equals $(p_i p_j/m)\times$ the Slutsky substitution effect of a marginal change in $p_j$ on $q_i$. Commodities $i$ and $j$ are then said to be Slutsky (or Hicksian, or $p-$) substititutes/complements when $s_{ij}$ is positive/negative. The substitution and income effects share the properties that:

$$(2) \quad \iota' b = 1,\ S = S',\ S\iota = 0, \text{ and } y'Sy < 0,\ \forall\ y : y \neq \theta\iota,\ \theta \text{ a real scalar},$$

where $\iota$ is the vector of units.

An inverse demand system expresses the consumer's willingness to spend on the different commodities in terms of the quantities consumed [3]. A measure of the willingness to spend on a commodity is provided by the normalised price of that commodity. It can be shown that in differential form, the Rotterdam parameterisation of the inverse demand system is very close to (1) (*see* e.g. BARTEN & BETTENDORF, 1989, eq 13):

$$(3) \qquad \hat{w}\,\mathrm{dln}\pi = g\left[w'\,\mathrm{dln}q\right] + H\,\mathrm{dln}q.$$

The vector $g$ is a vector of scale effects, measuring how the willingness to pay is affected by a change in the Divisia quantity index, while the $(i,j)$-element of the matrix $H$, $h_{ij}$, equals $q_i q_j\times$ the Antonelli substitution effect of a marginal change in consumption of commodity $j$ on the marginal willingness to spend on commodity $i$. When $h_{ij}$ takes on a positive/negative sign, commodities $i$ and $j$ are said to be Antonelli (or $q-$) complements/substitutes. These scale and substitution effects satisfy the following set of restrictions:

$$(4) \qquad \begin{gathered} \iota' g = -1,\ H = H',\ H\iota = 0, \\ \text{and } y'Hy < 0,\ \forall\ y : y \neq \theta\iota,\ \theta \text{ a real scalar}. \end{gathered}$$

3. See ANDERSON (1980) for an extensive discussion of inverse demand systems.

# 3 Indirect Allais Coefficients

Inserting the optimal commodity bundle $q(\pi)$ in the direct utility function yields the indirect utility function, $\nu(\pi)$. Because the direct utility function is strongly quasi concave, this indirect utility function will be strongly quasi convex [4]. Its vector of first derivatives is

$$(5) \qquad \nu_{\pi} =_{\text{def}} \frac{\partial \nu(\pi)}{\partial \pi} = -\lambda(\pi)\, q(\pi),$$

and thus $\pi' q = 1$ implies that $\pi' \nu_{\pi} = -\lambda$. Expression (5) suggests that the Hessian of the indirect utility function, $\nu_{\pi\pi}$, provides information on the way price changes affects outlays on the different commodities, when these outlays are all evaluated at the shadow price $\lambda$. Clearly, this information is not invariant to monotonous increasing transformations of the utility function. For instance, after transformation of the indirect utility function to $\tilde{\nu}(\pi) = f(\nu(\pi))$, $f'(\cdot) > 0$, its Hessian is transformed to $\tilde{\nu}_{\pi\pi} = f'' \nu_{\pi} \nu_{\pi}' + f' \nu_{\pi\pi}$. If $f(\cdot)$ is sufficiently concave, the transformation may change the sign of the Hessian elements.

We may, however, ordinalise this Hessian matrix in a way very similar to the ordinalisation proposed by Allais w.r.t. the direct utility function. Let us for this purpose select a standard pair of commodities, the pair $(r, s)$ say. Then it is not difficult to convince oneself that (subscripts with $\nu$ denote partial derivatives)

$$(6) \quad \pi' \tilde{\nu}_{\pi} \left( \frac{\tilde{\nu}_{ij}}{\tilde{\nu}_i \tilde{\nu}_j} - \frac{\tilde{\nu}_{rs}}{\tilde{\nu}_r \tilde{\nu}_s} \right) = f' \pi' \nu_{\pi} \left( \frac{\nu_{ij}}{f' \nu_i \nu_j} + \frac{f''}{f'^2} - \frac{\nu_{rs}}{f' \nu_r \nu_s} - \frac{f''}{f'^2} \right)$$
$$= \pi' \nu_{\pi} \left( \frac{\nu_{ij}}{\nu_i \nu_j} - \frac{\nu_{rs}}{\nu_r \nu_s} \right).$$

Let me therefore define the *indirect Allais interaction coefficient* between commodities $i$ and $j$ as

$$(7) \qquad c_{ij} =_{\text{def}} \eta(\pi) \frac{\nu_{ij}}{\nu_i \nu_j} - \beta(\pi),$$

where $\eta(\pi) =_{\text{def}} \pi' \nu_{\pi}$ and $\beta(\pi) =_{\text{def}} \eta(\pi) \cdot \nu_{rs} / (\nu_r \nu_s)$; the indirect Allais matrix is then defined as $C = (c_{ij})$. Before interpreting these interaction

4. Because preferences are *strictly* convex, and not just convex or even non-convex, the indirect utility function $\nu(\cdot)$ will be differentiable everywhere, or, to put it differently, the associated indifference curves in the space of normalized prices will not have “kinks”. Because $u(\cdot)$ is *strongly* quasi concave, the indifference surface in the space of normalized prices will not have “flat” regions (*see* DEATON & MUELLBAUER, 1980, pp. 47-9).

coefficients, note that with the help of this definition the second order partial derivative of the indirect utility function, $\nu_{ij}$, can be written as

$$(8) \qquad \nu_{ij} = \frac{1}{\eta(\pi)} \nu_i \nu_j c_{ij} + \frac{\beta(\pi)}{\eta(\pi)} \nu_i \nu_j .$$

Consider then the total differential of $(-\nu_i)$, the welfare loss a consumer experiences due to a marginal increase in $\pi_i$ : $\mathrm{d}(-\nu_i) = \Sigma_j (-\nu_{ij}) \mathrm{d}\pi_j$. Using (8), dividing through by $(-\nu_i)$ and exploiting the fact that $\eta(\pi) = -\lambda(\pi)$, one obtains

$$(9) \qquad \begin{aligned} \mathrm{d}\ln(-\nu_i) &= \Sigma_j \, c_{ij} \, w_j \, \mathrm{d}\ln \pi_j + \frac{\beta(\pi)}{\eta(\pi)} \Sigma_j \, \nu_j \, \mathrm{d}\,\pi_j \\ &= \Sigma_j \, c_{ij} \, w_j \, \mathrm{d}\ln \pi_j + \frac{\beta(\pi)}{\eta(\pi)} \mathrm{d}\nu. \end{aligned}$$

According to (9), the relative change in the utility loss of a marginal increase in $\pi_i$ due to a change in the normalised price $\pi_j$ is controlled by a commodity specific effect and a general effect. The former is ordinal and equals the indirect Allais interaction coefficient $c_{ij}$ weighted by the importance of commodity $j$ in the consumer's budget, $w_j$. The general effect has a cardinal nature, and since $(\beta/\eta) \cdot \mathrm{d}\nu = \beta \Sigma_j \, w_j \, \mathrm{d}\ln \pi_j$, this effect works through the impact of the price change of commodity $j$ on the Divisia normalised price index. By an appropriate change in this index, utility is kept constant, and the elasticity of $\nu_j$ w.r.t. $\pi_j$ reduces to $c_{ij} w_j$.

The fact that a consumer's welfare loss due to a marginal increase of commodity $i$'s normalised price equals $\lambda q_i$ suggests the following classification rule: if $c_{ij} > 0 \, (< 0)$ commodity $i$ is a substitute for (complement of) commodity $j$ and if $c_{ij} = 0$ the two commodities are independent; here "substitute/complement for" should be read as "a better substitute/complement for ... than commodity $r$ is for commodity $s$". It also implies that $\partial \ln(q_i/q_j)/\partial \ln \pi_k = (c_{ik} - c_{jk)\, w_k}$: the difference between the uncompensated price elasticities of commodities $i$ and $j$ w.r.t. $\pi_k$, is a fraction of the difference in the indirect Allais interactions $(i, k)$ and $(j, k)$, the fraction being the importance of commodity $k$ in the budget.

There are some theoretical properties which apply on the indirect Allais matrix [5]:

$$(10) \qquad C = C', \text{ and } y'Cy < 0, \; \forall \, y : \iota' y = 0.$$

5. Because $C = \eta(\pi) \, \hat{\nu}_\pi^{-1} \nu_{\pi\pi} \hat{\nu}_\pi^{-1} - \beta(\pi) \, \iota\iota'$, the first property follows from the symmetry of $\nu_{\pi\pi}$. Moreover, strong quasi convexity of the indirect utility function $\nu(\pi)$, $x' \nu_{\pi\pi} x > 0, \forall \, x :$ $x' \nu_\pi = 0$, implies the second property (choose $y = \nu_\pi' x$) since $\eta(\pi) = \pi' \nu_\pi = -\lambda(\pi) < 0$.

The symmetry of $C$ immediately follows from the symmetry of $\nu_{\pi\pi}$. The second property looks like a negativity property but is much weaker [6]. In particular, the negativity of the diagonal elements is not implied because the elements of the unit vector (0, ..., 1, ..., ...0)′ do not add up to zero. There is therefore no *a priori* reason for the indirect Allais classification to favour substitutability [7].

Writing (9) in matrix notation gives rise to a new regular demand system:

$$\mathrm{d}\ln(-\nu_\pi) = \beta\iota\,[w'\,\mathrm{d}\ln\pi] + C\hat{w}\,\mathrm{d}\ln\pi \tag{11}$$

Although the LHS variables of this system are unobservable, it would be wrong to conclude that the indirect Allais matrix cannot be inferred from market behaviour. The following proposition indeed shows that the indirect Allais matrix can be related to the parameters of the regular and inverse Rotterdam demand system, parameters which can be empirically determined. (Proofs of propositions are relegated to the paper's appendix.)

PROPOSITION 1 : The relation between the indirect Allais matrix and the parameters of the regular and inverse Rotterdam demand systems is as follows:

$$\begin{pmatrix} S & b \\ b' & 0 \end{pmatrix} = \begin{pmatrix} -\Omega & w \\ w' & 1 \end{pmatrix} \begin{pmatrix} C - \rho_C\,\iota\iota' & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} -\Omega & w \\ w' & 1 \end{pmatrix}, \tag{12}$$

$$\begin{pmatrix} C - \frac{1}{\varphi_C}\,\iota\iota' & -\iota \\ -\iota' & 0 \end{pmatrix} \begin{pmatrix} H & g \\ g' & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}. \tag{13}$$

where $\rho_C =_{\mathrm{def}} 2 + w'Cw$, $\varphi_C =_{\mathrm{def}} \iota' C^{-1}\iota$ and $\Omega =_{\mathrm{def}} (\hat{w} - ww')$.

Inspection of (13) learns that the indirect Allais matrix is a generalised inverse of the Antonelli matrix $H$: by multiplying out the LHS one obtains $CH - \iota g' = I$; premultiplying this relation with $H$ then yields $HCH = I$ since $H\iota = 0$. It is also clear that the Rotterdam parameterisation of the inverse demand system is consistent with constant indirect Allais coefficients.

6. Indeed, the set of vectors for which pre- and postmultiplication of the matrices $S$ and $H$ will results in a negative number is given by $Y_1 =_{\mathrm{def}} \{y | y$ not proportional with $\iota\}$. For the matrix $C$, this set is given by $Y_2 =_{\mathrm{def}} \{y | y'\iota = 0\}$. Compared with $Y_1$, $Y_2$ is infinitesimally small.

7. One could argue that the dominance of substitutability can be removed from the Slutsky classification by considering $s_{ij} - s_{rs}$. The reason for subtracting from $\pi'\nu_\pi\nu_{ij}/(\nu_i\nu_j)$ the corresponding expression for the standard pair is to remove the cardinality (*see* (6)). The Slutsky coefficients, however, are already ordinal. It is therefore not clear what the transformation $s_{ij} - s_{rs}$ precisely means.

The relation between the indirect Allais matrix and the Slutsky and income effects is given by expression (12). Since $\begin{pmatrix} -\Omega & w \\ w' & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\hat{w}^{-1} & \iota \\ \iota' & 0 \end{pmatrix}$, an equivalent way of writing this expression is

$$(14) \qquad \begin{pmatrix} C - \rho_C \iota\iota' & \iota \\ \iota' & 0 \end{pmatrix} = \begin{pmatrix} -\hat{w}^{-1} & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} S & b \\ b' & 0 \end{pmatrix} \begin{pmatrix} -\hat{w}^{-1} & \iota \\ \iota' & 0 \end{pmatrix},$$

In particular,

$$(15) \qquad C - \rho_C \iota\iota' = \hat{w}^{-1} S \hat{w}^{-1} - \iota b' \hat{w}^{-1} - \hat{w}^{-1} b \iota'.$$

Thus, the Rotterdam parameterisation of a regular demand system will in general imply a variable indirect Allais matrix. From (15), the positive relationship between Slutsky and indirect Allais also transpires; strong Slutsky effects will give rise to strong indirect Allais interactions.

The parameters of either the regular or inverse Rotterdam demand system determine the indirect Allais matrix up to a constant ($\varphi_C$ and $\rho_C$). This indeterminacy is related to the selection of the standard pair. Once the indirect Allais interaction coefficient for the standard pair is fixed, the remaining interaction coefficients are fully determined. For instance, if estimates are available for $S$ and $b$, one first solves for $\rho_C$ by constraining $c_{rs} = 0\,(\rho_C = -s_{rs}/(w_r w_s) + b_r/w_r + b_s/w_s)$; next one adds this value to each element of the RHS matrix in (15). And in an analogous way the estimates of the inverse demand system can be used to compute the indirect Allais interaction coefficients by postmultiplying (13) through by the inverse of the second LHS matrix, look for the value of $1/\varphi_C$ for which $c_{rs} = 0$, and add this value to every element of the NW-block of this inverse matrix.

I will now relate the indirect Allais interaction coefficients to those originally suggested by ALLAIS (1943, p. 147). The latter are based on the ordinalised Hessian of the direct utility function. In particular, the *direct* Allais interaction coefficient between commodities $i$ and $j$, is defined as (subscripts with $u$ denote partial derivatives):

$$(16) \qquad a_{ij} = q' u_q \left( \frac{u_{ij}}{u_i u_j} - \frac{u_{rs}}{u_r u_s} \right).$$

Again, these coefficients are invariant to any monotone increasing transformation of the utility function and therefore reflect properties of the preference ordering. A positive coefficient is associated with complementarity and a negative one with substitutability; if the coefficient is zero, commodities $i$ and $j$ are said to be independent. If the *direct Allais matrix* is then defined as the $n \times n$ symmetric matrix $A =_{\text{def}} (a_{ij})$, its relation w.r.t. the indirect Allais matrix is as given by the following proposition:

PROPOSITION 2 : The direct and indirect Allais matrices relate in the following way:

$$(17)\quad \begin{pmatrix} A - \frac{1}{\varphi_A}\iota\iota' & \iota \\ \iota' & 0 \end{pmatrix} = \begin{pmatrix} -\hat{w}^{-1} & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} C - \rho_C \iota\iota' & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} -\hat{w}^{-1} & \iota \\ \iota' & 0 \end{pmatrix},$$

$$(18)\quad \begin{pmatrix} C - \frac{1}{\varphi_C}\iota\iota' & \iota \\ \iota' & 0 \end{pmatrix} = \begin{pmatrix} -\hat{w}^{-1} & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} A - \rho_A \iota\iota' & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} -\hat{w}^{-1} & \iota \\ \iota' & 0 \end{pmatrix}.$$

Direct and indirect Allais matrices are therefore inversely related to one another [8]. That this inverse relationship is blurred by the vector of budget shares is also intuitive: a constant direct Allais matrix is consistent with the Rotterdam parameterisation of a *regular* demand system, but not with that of an *inverse* demand system (with which a constant indirect Allais matrix is consistent). Simple but tedious matrix algebra allows to rewrite expressions (17) and (18) as

$$(19)\quad A - \frac{1}{\varphi_A}\iota\iota' = \hat{w}^{-1}\left[C^{-1} + \rho_C ww' - \frac{1}{\varphi_C}(C^{-1}\iota + w)(C^{-1}\iota + w)'\right]\hat{w}^{-1},$$

$$(20)\quad C - \frac{1}{\varphi_C}\iota\iota' = \hat{w}^{-1}\left[A^{-1} + \rho_A ww' - \frac{1}{\varphi_A}(A^{-1}\iota + w)(A^{-1}\iota + w)'\right]\hat{w}^{-1}$$

8. This is not surprising since the direct Allais matrix relates inversely to the Slutsku matrix but positively to the Antonelli matrix:

$$\begin{pmatrix} A - \frac{1}{\varphi_A}\iota\iota' & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} S & b \\ b' & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} H & g \\ g' & 0 \end{pmatrix} = \begin{pmatrix} \Omega & -w \\ w' & 1 \end{pmatrix} \begin{pmatrix} A - \rho_A \iota\iota' & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} \Omega & w \\ -w' & 1 \end{pmatrix},$$

where $\varphi_A =_{\mathrm{def}} \iota' A^{-1}\iota$, $\rho_A =_{\mathrm{def}} 2 + w' A w$. *See* BARTEN (1990). BARTEN & BETTENDORF (1989) explore the direct Allais coefficients implied by an inverse demand system for fish.

(*see* Appendix). Thus the same matrix function which maps $A$ into $C$ also maps $C$ into $A$. The latter pair of expressions may be used to trace out the implications of certain assumptions on the interaction matrix of one type for the interactions of the other type. In the working paper (SCHROYEN, 1995), I have shown that weak and strong indirect separability assumptions place a simplifying structure on the indirect Allais matrix and explore by means of (19) and (12) the implications of such assumptions for the direct Allais and Antonelli matrices.

# 4 Discussion and interpretation

I have now constructed a new ordinal classification rule based on the Hessian of the indirect utility function and have related this rule to the Slutsky, Antonelli and direct Allais classifications. All these classifications have exactly the same empirical content, and one may deliberate on their distinction and relative usefulness.

I would like to offer the interpretation that the differences among the different classifications arise from different normalisations. The (Rotterdam parameterisation of) Slutsky and Antonelli coefficients are normalised in the sense that for each commodity $i$, these coefficients should have a zero average, e.g. $\Sigma_j\, s_{ij} = 0$. This normalisation follows from the linear homogeneity of the expenditure/distance function in prices/quantities. The concavity of these two functions in their arguments implies that $s_{ii}$ and $h_{ii}$ are non-positive, and therefore that at least one of the off-diagonal interaction coefficients will bear a positive sign. When such a sign is interpreted as indicating a substitutability under Slutsky (complementarity under Antonelli), the Slutsky/Antonelli classification will exhibit a dominance of substitutability/complementarity.

Both Allais classifications are obtained after normalising the interaction coefficient of a standard pair to zero (or to some other figure, for that matter). Apart from the “weak negativity” property of the type described by the second part of (9), no other restrictions apply. Thus fewer *a priori* restrictions to comply with are bought at the cost of having to provide *a priori* information on the interaction among a standard pair of commodities. The trade-off is clear: either one goes for the Slutsky/Antonelli normalisations, accepting that this gives rise to a dominance of one type of interaction in interpretation, or one goes for the one of the “dominance free” Allais classifications, but chooses the normalisation oneself (through selection of the standard pair). This choice could be influenced by computational considerations: while Slutsky coefficients and their standard errors are readily available when estimating a regular demand system, either of the Allais interaction coefficients can only be obtained through computation (the standard errors of these coefficients can also be obtained by Monte Carlo methods or by the linear approximation rule proposed by KLEIN [1953], p. 258); however, the availability of matrix routines in modern econometric software weakens this argument.

# 5 Complements and Substitutes in Foodstuff and Beverages Consumed by the Beguinage of Lier in the 16th Century

Some years ago, SCHOKKAERT & VAN DER WEE (1988, S & W hereafter) carried out an econometric study of the consumption of foodstuff by the infirmary of the beguinage of Lier (a town 25 km South-East of Antwerp in Belgium) during the 16th century. Disposing of the accounts of the infirmary for the period 1526-1575, and of excellent data on prices for ten food and beverage items for the same period, the authors described the allocation for foodstuffs by estimating a regular Rotterdam demand system [9]. From the point of view of classifying commodities, this study is inviting as one would expect that among ten items of food and beverages both strong interactions of substitutability as well as complementarity arise. Moreover, it is possible to verify such interactions with one's intuition and knowledge about eating and cooking habits–something which is much harder when trying to explain the interactions among, say, the ten main consumption categories of the national accounts.

The ten foodstuff categories studied by S & W, their average budget shares and the estimation results for the marginal budget shares and Slutsky coefficients are provided in table 1. Note that "Spices" means "spices and sugar", "Vinegar" means "vinegar and oil", and "Cheese" stands for "cheese and butter". The large average share of beer in the Infirmary's budget should not worry the reader: it concerned a very light hop beer! For a detailed description of all categories, the reader is referred to the S & W study.

Using these results, I have computed the implied Antonelli, direct and indirect Allais coefficients. For the latter, I have chosen beer and oil as the standard pair, mainly because one would regard these two commodities neither as strong substitutes nor as strong complements. These computations are reported elsewhere (*see* SCHROYEN, 1995).

9. For several reasons, the estimation of this allocation process is of high quality. First, because the price data show a lot of variation over the period (mainly due to the temporary occurrence of severe shortages), a huge number of coefficients have been estimated with relatively high precision. Second, because on average no more than ten nuns inhabited the infirmary, aggregation problems may be deemed insignificant, if not absent.

TABLE 1

***Estimation results of the regular Rotterdam model (1526-1575).***

| | S | | | | | | | | | | b | $R^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Wheat | Rye | Beer | Wine | Fish | Spice | Oil | Salt | Meat | Cheese | | |
| Wheat | –0.726** | .0166** | .0191* | .0021 | .0002 | –.0018 | .0076 | .0012 | .0162 | .0114* | .0911** | .796 |
| (.075) | (.0157) | (.0148) | (.0136) | (.0066) | (.0066) | (.0119) | (.0038) | (.0043) | (.0183) | (.0071) | (.0327) | |
| Rye | | –.1059** | .0416* | .0067* | .0034 | –.0026 | –.0083** | .0037 | .0569* | –.0122* | .2801** | .546 |
| (.37) | | (.0399) | (.0245) | (.0062) | (.0081) | (.0129) | (.0036) | (.0044) | (.0299) | (.0067) | (.0684) | |
| Beer | | | –.1631** | .0040 | .0123* | .0218* | .0057* | –.0039 | .0512* | .0114* | .1821** | .667 |
| (.19) | | | (.0303) | (.0067) | (.0084) | (.0139) | (.0038) | (.0048) | (.0287) | (.0073) | (.0423) | |
| Wine | | | | –.0151* | .0073* | –.0014 | .0020 | –.0040* | –.0114 | .0098* | .0315** | .423 |
| (.015) | | | | (.0083) | (.0038) | (.0074) | (.0033) | (.0029) | (.0123) | (.0054) | (.0139) | |
| Fish | | | | | –.0312** | .0058 | –.0020 | –.0004 | .0224* | –.0178* | .0537** | .545 |
| (.037) | | | | | (.0055) | (.0072) | (.0021) | (.0025) | (.0115) | (.0038) | (.0164) | |
| Spices[a] | | | | | | –.0455** | –.0047* | .0096* | .0242* | –.0064 | .0368* | .217 |
| (.045) | | | | | | (.0180) | (.0043) | (.0049) | (.0230) | (.0076) | (.0271) | |
| Oil[b] | | | | | | | –.0073** | .0054** | .0113* | –.0097** | .0102* | .421 |
| (.009) | | | | | | | (.0025) | (.0017) | (.0062) | (.0030) | (.0080) | |
| Salt | | | | | | | | –.0063** | –.0128* | .0074** | .0099* | .206 |
| (.012) | | | | | | | | (.0025) | (.0075) | (.0029) | (.0096) | |
| Meat | | | | | | | | | –.1864** | .0285** | .286** | .433 |
| (.237) | | | | | | | | | (.0479) | (.0112) | (.061) | |
| Cheese[c] | | | | | | | | | | –.0223** | .0184 | .495 |
| (.01) | | | | | | | | | | (.0071) | (.0192) | |

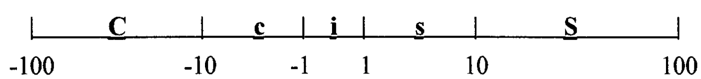
*Average budget shares are given in parentheses in the first column. Figures in parentheses in the next columns are standard errors. Coefficients were estimated under the restrictions set out in (2) of the text. A single(*)/double(**) asterisk with a coefficient denotes that it is larger than once/twice its standard error. [a] Spices & sugar; [b] Oil & vinegar; [c] Cheese & butter.*
*Source: Schokkaert & Van der Wee (1988, p. 147).*

With 10 commodities, the number of interactions among distinct commodities amounts to 45. It is not my intention here to discuss how each of these interactions are treated under every classification rule. Rather, I want to present some summary statistics and to comment on some commonalities and striking discrepancies across the classifications.

To allow comparison across classifications, it is necessary to normalise the computed interaction coefficients. One possibility would be to express the interactions as elasticities [10]; these are dimensionless and a standard rule is to say than an elasticity is high (low) when in absolute value it exceeds (falls short of) unity. But elasticities have the undesirable property that they are not symmetric. Another way of normalisation was put forward by ALLAIS [1943] himself. He suggested to transform $a_{ij}$ into $a_{ij}/\sqrt{(a_{ii}a_{jj})}$. But the possibility to carry out this normalisation may depend on the selection of the standard pair. Indeed, if the selection of that pair results in not all diagonal elements having a negative sign, this normalisation is not defined [11]. So I shall have recourse to a third normalisation procedure which is at the same time symmetric, dimensionless and always feasible. First, I calculate for each matrix the difference between the largest and smallest off-diagonal element. Next, I express all off-diagonal elements as a percentage of this difference. For instance, the normalised Slutsky effect between commodities $i$ and $j\,(j \neq i)$ is defined as

$$\frac{s_{ij}}{M_S - m_S} \cdot 100, \tag{21}$$

where $M_S = \max\{s_{ij} | i \neq j\}$ and $m_s = \min\{s_{ij} | i \neq j\}$. In the same way, I normalise the off-diagonal elements of $C, A$ and $H$, except that for the latter two the sign is reversed; in that case a positive (negative) sign indicates substitutability (complementarity) under all four classifications. I then use the following scale to label interactions:




where **C(S)** indicates strong complementarity (substitutability), **c(s)** indicates weak complementarity (substitutability), and **i** indicates independence.

The interaction coefficients thus obtained and the classifications implied are displayed in table 2.

10. This would require division of $s_{ij}$ and $h_{ij}$ by $w_j$ and multiplication of $a_{ij}$ and $c_{ij}$ by $w_j$.

11. Recall that negativity of the diagonal elements of $C$ is not required. The same is true for the direct Allais matrix $A$. In the present case, all $c_{ii}\ (i = 1, \ldots, 10)$ are negative, so $\sqrt{(c_{ii}c_{jj})}$ is defined for all $i, j$. On the other hand, the 3rd and 9th diagonal elements of the direct Allais matrix are positive because element (3,7) of $\begin{pmatrix} S & b \\ b' & 0 \end{pmatrix}^{-1}$ is more negative than the 3rd and 9th diagonal elements of this matrix–see the first expression of footnote 8.

TABLE 2

***Normalised interaction coefficients.***

| Slutsky | Wheat | Rye | Beer | Wine | Fish | Spices | Oil | Salt | Meat | Cheese |
|---|---|---|---|---|---|---|---|---|---|---|
| Wheat | | 22.22 | 25.57 | 2.81 | 0.27 | –2.41 | 10.17 | 1.61 | 21.69 | 15.26 |
| Rye | S | | 55.69 | 8.97 | 4.55 | –3.48 | –11.11 | 4.95 | 76.17 | –16.33 |
| Beer | S | S | | 5.35 | 16.47 | 29.18 | 7.63 | –5.22 | 68.54 | 15.26 |
| Wine | s | s | s | | 9.77 | –1.87 | 2.68 | –5.35 | –15.26 | 13.12 |
| Fish | i | s | S | s | | 7.76 | –2.68 | –0.54 | 29.99 | –23.83 |
| Spices | c | c | S | c | s | | –6.29 | 12.85 | 32.40 | –8.57 |
| Oil | S | C | s | s | c | c | | 7.23 | 15.13 | –12.99 |
| Salt | s | s | c | c | i | S | s | | –17.14 | 9.91 |
| Meat | S | S | S | C | S | S | S | C | | 38.15 |
| Cheese | S | C | S | S | C | c | C | s | S | |

| Anton. | Wheat | Rye | Beer | Wine | Fish | Spices | Oil | Salt | Meat | Cheese |
|---|---|---|---|---|---|---|---|---|---|---|
| Wheat | | –32.88 | 0.61 | –2.69 | –22.02 | 7.09 | 5.51 | 23.9 | –3.43 | 8.76 |
| Rye | C | | –1.31 | 6.17 | 47.5 | –15.87 | –12.54 | –52.5 | 7.64 | –18.74 |
| Beer | i | c | | –0.04 | –0.098 | 0.21 | .010 | 0.85 | –0.07 | 0.43 |
| Wine | c | s | i | | 3.75 | –1.45 | –1.25 | –4.51 | 0.77 | –1.40 |
| Fish | C | S | i | s | | –10.15 | –7.70 | –34.59 | 4.78 | –13.00 |
| Spices | s | C | i | c | C | | 2.90 | 11.55 | –1.82 | 3.93 |
| Oil | s | C | i | c | c | s | | 9.14 | –1.55 | 2.90 |
| Salt | S | C | i | c | C | S | s | | –5.70 | 13.62 |
| Meat | c | s | i | i | s | c | c | c | | –1.79 |
| Cheese | s | C | i | c | C | s | s | S | c | |

| Dir All | Wheat | Rye | Beer | Wine | Fish | Spices | Oil | Salt | Meat | Cheese |
|---|---|---|---|---|---|---|---|---|---|---|
| Wheat | | –0.66 | –0.02 | –1.3 | –4.17 | 1.06 | 4.23 | 13.81 | –0.14 | 6.03 |
| Rye | i | | –0.05 | 0.54 | 1.79 | –0.54 | –2.00 | –6.24 | 0.01 | –2.71 |
| Beer | i | i | | –0.05 | –0.10 | –0.03 | 0 | 0.13 | –0.04 | 0.05 |
| Wine | c | i | i | | 3.51 | –1.17 | –4.87 | –13.17 | 0.07 | –4.95 |
| Fish | c | s | i | s | | –3.21 | –12.1 | –40.76 | 0.26 | –18.41 |
| Spices | s | i | i | c | c | | 3.72 | 11.11 | –0.13 | 4.50 |
| Oil | s | c | i | c | C | s | | 44.17 | –0.41 | 16.77 |
| Salt | S | c | i | C | C | S | S | | –1.11 | 59.24 |
| Meat | i | i | i | i | i | i | i | c | | –0.47 |
| Cheese | s | c | i | c | C | s | S | S | i | |

| Ind All | Wheat | Rye | Beer | Wine | Fish | Spices | Oil | Salt | Meat | Cheese |
|---|---|---|---|---|---|---|---|---|---|---|
| Wheat | | –1.52 | –1.20 | –1.56 | –2.23 | –2.21 | 4.46 | –1.14 | –1.60 | 6.33 |
| Rye | c | | –1.37 | –1.68 | –1.86 | –1.73 | –3.27 | –1.17 | –1.49 | –4.15 |
| Beer | c | c | | –1.68 | –1.10 | –0.27 | 0 | –2.76 | –1.32 | 1.14 |
| Wine | c | c | c | | 4.86 | –3.62 | 6.01 | –15.34 | –4.50 | 34.94 |
| Fish | c | c | c | s | | –0.02 | –5.71 | –2.58 | –0.78 | –30.58 |
| Spices | c | c | i | c | i | | –8.60 | 8.64 | –0.58 | –10.53 |
| Oil | s | c | i | s | c | C | | 27.18 | 1.00 | –65.06 |
| Salt | c | c | c | C | c | s | S | | –4.53 | 33.54 |
| Meat | c | c | c | c | i | i | i | c | | 4.49 |
| Cheese | s | c | s | S | C | C | C | S | s | |

Close inspection of table 2 reveals several regularities; these will now be discussed.

A first feature is that the number of weak and strong substitutabilities is the highest under the Slutsky criterion (29 or 65% out of 45), lowest under indirect Allais (10 or 22%), and in between under Antonelli (15 or 33.3%) and direct Allais (12 or 26.7%). See table 3. This illustrates the claim that indirect Allais is less liable to favour substitutability than Slutsky.

TABLE 3

***Frequency of interactions.***

| | Slutsky | Antonelli | Dir Allais | Ind Allais |
|---|---|---|---|---|
| Substitutability | .65 | .33 | .27 | .23 |
| strong | .38 | .09 | .11 | .07 |
| weak | .27 | .24 | .16 | .16 |
| Independence | .04 | .20 | .42 | .13 |
| Complementarity | .31 | .47 | .31 | .64 |
| strong | .13 | .20 | .09 | .11 |
| weak | .18 | .27 | .22 | .53 |

To gauge the similarity in classifying commodities by the four classification rules, I have computed the ordinary and rank correlation coefficients between the off-diagonal normalised matrix elements; these correlations are reported in table 4 above and below the diagonal, respectively [12]. Despite the inverse relationship which each matrix has w.r.t. two others, none of the correlation coefficients are negative, which is reassuring since that would mean that exploiting different information about the same preference ordering results in a totally opposite classification of commodities. The highest rank correlation occurs between direct Allais and Antonelli (.91), followed by Slutsky and indirect Allais (.70). Again, Slutsky and direct Allais differ widely (.18). The ordinary correlation coefficients take into account both the relative position of the interactions and the intensity with which these interactions differ [13]. Except for Dir Allais-Ind Allais, the ordinary correlation coefficients are all below the corresponding rank correlation coefficients. This means, for instance, that even though Antonelli and direct Allais rank pairs of commodities on the complements-

12. For this purpose, I considered the four sets of 45 interaction coefficients as four data series. Of course, these four data series are, through estimation and derivation, obtained using the same sample of market data, and as such are correlated through sampling errors. The correlation coefficients reported in Table 4 should therefore be taken as purely indicative.

13. Note that the rankings of the interactions within both the direct and indirect Allais classification is independent of the selection of the standard pair: selecting a new standard pair means subtracting the coefficient pertaining to that pair of the old interaction matrix. This is a monotone transformation (a linear one). None of the rank correlation coefficients is therefore affected by the choice of a different standard pair; not do the ordinary correlations change.

substitutes spectrum in a very similar way, there is much less agreement w.r.t. to the intensities of pair being complements/substitutes.

TABLE 4

***Ordinary (above diagonal) and rank (below diagonal) correlation coefficients among the different classifications.***

| | Slutsky | Antonelli | Dir Allais | Ind Allais |
|---|---|---|---|---|
| Slutsky | | .14 | .07 | .28 |
| Antonelli | .20 | | .48 | .13 |
| Dir Allais | .18 | .91 | | .34 |
| Ind Allais | .70 | .32 | .27 | |

I shall now comment on some specific interactions. In particular, I want to draw attention on some striking similarities between the four classification rules, and also on some striking discrepancies between the Slutsky classification on the one hand and the three rules on the other.

There are 13 pairs of commodities for which the type of interaction is robust w.r.t. the classification rule. Thus 29% of the interactions are under all four criteria are classified either as substitutes or as complements; they are listed in table 5, along with their degree of interaction. According to all four classifications, salt interacts with spices, oild and cheese as a substitute, and with wine and meat as a complement. “Salt”, S & W (p. 143) write, “was an important ingredient in the kitchen of the Middle Ages and the Early Modern Times. Salting perishable foodstuffs such as meat, fish and butter was the most common way at that time to preserve them.” An they go on to argue that “Spices, including sugar, were used as drugs, also as flavourings, and as preservatives [...]” (S & W, p. 142). This explains the substitutability of spices with salt. Recall that the category “oil” comprises both oil and vinegar. The use of the latter for flavouring purposes explains the substitution possibilities with salt. The fact that the category cheese consists of both cheese and butter and that butter was salted in those days for preservation purposes, nicely explains the strong degree of substitutability among salf and the category cheese. An odd relationship is the complementarity between salt and wine, which is even quite strong according to both Allais classifications. Wheat and cheese being substitutes (although not in a very strong way) looks peculiar, but probably this is due to the fact that both white bread, cheese and butter were regarded as luxury products in the 16 th century: two luxury products having to fit into the same budget may give rise to substitutability, even though from a dietary point of view they are complementary. Whence, if cheese and/or butter were consumed, this was probably in combination with the much cheaper rye bread.

TABLE 5

***Robust interactions.***

| Robust substitutes | Slutsky | Antonelli | Dir Allais | Ind Allais |
|---|---|---|---|---|
| wheat & oil | 10.47 | 5.51 | 4.23 | 4.46 |
| wheat & cheese | 15.26 | 8.76 | 6.03 | 6.33 |
| fish & wine | 9.77 | 3.75 | 3.51 | 4.86 |
| salt & spices | 12.85 | 11.55 | 11.11 | 8.64 |
| salt & oil | 7.23 | 9.14 | 44.17 | 27.18 |
| salt & cheese | 9.91 | 13.62 | 59.24 | 33.54 |
| **Robust complements** | **Slutsky** | **Antonelli** | **Dir Allais** | **Ind Allais** |
| oil & rye | –11.11 | –12.54 | –2 | –3.27 |
| cheese & rye | –16.33 | –18.74 | –2.71 | –4.15 |
| spices & wine | –1.87 | –1.45 | –1.17 | –3.62 |
| salt & wine | –5.35 | –4.51 | –13.17 | –15.34 |
| fish & oil | –2.68 | –7.7 | –12.1 | –5.71 |
| fish & cheese | –23.83 | –13 | –18.41 | –30 |
| salt & meat | –17.14 | –5.7 | –1.11 | –4.53 |

Finally, I have checked for which pairs of commodities the Slutsky criterion implies an interaction which is different from the one implied by the other three criteria–*see* table 6. Deviant Slutsky substitutes are: wheat & wine, salt & rye, spices & beer. The first two of these pairs are classified as (weak) complements under the other three classifications. The use of both salt and rye to bake rye bread would indeed suggest these items to be complements. Spices & beer are strong substitutes according to Slutsky, while the other three criteria regard these items as independent from one another, which is also more in line with intuition. Deviant Slutsky independents are fish & wheat, and fish & salt. The other three criteria classify both pairs as complements, which is more likely in view of the preservative role of salt. Deviant Slutsky complements are not encountered, but then such complements are less likely to occur than deviant substitutes because of the inherent dominance of substitutability.

TABLE 6

***Deviant Slutsky classifications.***

| Deviant Slutsky substitutes | Slutsky | Antonelli | Dir Allais | Ind Allais |
|---|---|---|---|---|
| wheat & wine | 2.81 | –2.69 | –1.3 | –1.56 |
| salt & rye | 1.61 | –52.5 | –6.24 | –1.17 |
| spices & beer | 29.18 | 0.21 | –.03 | –.027 |
| **Deviant Slutsky independents** | **Slutsky** | **Antonelli** | **Dir Allais** | **Ind Allais** |
| wheat & fish | 0.27 | –22.02 | –4.17 | –2.21 |
| fish & salt | –0.54 | –34.59 | –40.76 | –2.58 |

# 6 Concluding Remarks

In this paper, I have considered the transformed Hessian of the indirect utility function as a fourth source to make inference about the way commodities interact with one another at the preference level. I have coined this the indirect Allais matrix because it results from the same type of ordinalisation procedure which ALLAIS [1943] applied on the Hessian of the direct utility function. I have showed that the indirect Allais is positively related to the Slutsky matrix, but inversely to the Antonelli and direct Allais matrices.

Indirect Allais coefficients can be given the interpretation that they measure the compensated elasticity of the marginal welfare loss a consumer experiences due to a price rise of one commodity w.r.t. a price rise of another commodity. A positive coefficient is then associated with substitutability, while a negative coefficient indicates complementarity. I have argued that the indirect Allais classification does not favour substitutability to the extent that the Slutsky classification does. This was confirmed by inspection of an empirical demand system for foodstuffs and beverages.

A weakness of the indirect Allais criterion is clearly the dependence on the selected standard pair–a wheakness shared by the direct Allais classification. One can look upon this in two ways. Either, one keeps firmly in mind that “$i$ and $j$ are complementary/substitutable” really means “more complementary/substitutable than $r$ and $s$ are”. Or, if one is prepared to take a more Bayesian approach to empirical demand analysis, one can think of both Allais classifications as inviting for some extra information to be provided by the empirical researcher. Irrespective of the interpretation, I believe that the indirect Allais matrix reveals properties of the preference ordering which are not provided by either of the three other classification rules.

## APPENDIX

*Proof of Proposition 1*:

To obtain expression (12), one can start from the "demand system" (11) and replace the LHS by $\mathrm{dln}\,\lambda\iota + \mathrm{dln}\,q$ (*see* (5)). Rearranging and multiplying through by $\hat{w}$ then yields:

$$(A1) \qquad \hat{w}\,\mathrm{dln}\,q = \hat{w}C\hat{w}\,\mathrm{dln}\,\pi + w\,(\beta w'\,\mathrm{dln}\,\pi - \mathrm{dln}\,\lambda).$$

Premultiplying through by $\iota'$, and rearranging produces

$$(A2) \qquad (\beta w'\,\mathrm{dln}\,\pi - \mathrm{dln}\,\lambda) = w'\,\mathrm{dln}\,q - w'C\hat{w}\,\mathrm{dln}\,\pi.$$

Substituting the round brackets term in (A1) for the RHS of (A2) results in

$$(A3) \qquad \hat{w}\,\mathrm{dln}\,q = ww'\,\mathrm{dln}\,q + \Omega C\hat{w}\,\mathrm{dln}\,\pi.$$

where $\Omega =_{\mathrm{def}} \hat{w} - ww'$. Adding and substracting $bw'\,\mathrm{dln}\,\pi$ to the RHS, making use of the fact that $w'\,\mathrm{dln}\,q = -w'\,\mathrm{dln}\,\pi$ and rearranging, one obtains

$$(A4) \qquad \hat{w}\,\mathrm{dln}\,q = -bw'\,\mathrm{dln}\,\pi + (\Omega C\hat{w} - ww' + bw')\,\mathrm{dln}\,\pi.$$

But since in differential form, the Rotterdam parameterisation of the regular demand system $q(\pi)$ is given by (1), the matrix in round brackets with which dln $\pi$ is premultiplied on the RHS in (A4) must equal the matrix $S$. And since $S\iota = 0$, if follows that $b = (I - \Omega C)\,w$ and therefore that $S = \Omega C\Omega$. These relationships can then be collected as in (12).

To see how the $C$ matrix relates to the Antonelli matrix, one premultiplies (A1) through by $C^{-1}\hat{w}^{-1}$ and isolates $\hat{w}\,\mathrm{dln}\,\pi$:

$$(A5) \qquad \hat{w}\,\mathrm{dln}\,\pi = C^{-1}\,\mathrm{dln}\,q - C^{-1}\iota\,(\beta w'\,\mathrm{dln}\,\pi - \mathrm{dln}\,\lambda).$$

Premultiplying through by $\iota'$, the round brackets term can again be isolated:

$$(A6) \qquad (\beta w'\,\mathrm{dln}\,\pi - \mathrm{dln}\,\lambda) = (\iota C^{-1}\iota)^{-1}(\iota' C^{-1}\,\mathrm{dln}\,q - w'\,\mathrm{dln}\,\pi).$$

Taking advantage of this equality, another way of writing (A5) is then

$$(A7) \qquad \hat{w}\,\mathrm{dln}\,\pi = [C^{-1} - C^{-1}\iota\,(\iota'C^{-1}\iota)^{-1}\,\iota'C^{-1}]\,\mathrm{dln}\,q + C^{-1}\iota\,(\iota'C^{-1}\iota)^{-1}\,w'\,\mathrm{dln}\,\pi.$$

But since $w'\,\mathrm{dln}\,\pi = -w'\,\mathrm{dln}\,q$, this expression can also be written as

$$(A8) \quad \hat{w}\,\mathrm{dln}\,\pi = \left[-\frac{1}{\varphi_C}\,C^{-1}\iota\right] w'\,\mathrm{dln}\,q + \left[C^{-1} - \frac{1}{\varphi_C}\,C^{-1}\iota\iota'C^{-1}\right]\mathrm{dln}\,q.$$

where $\varphi_C$ is a shorthand for $\iota'C^{-1}\iota$.

In view of the Rotterdam parameterisation of the inverse demand system (*see* (3)), it is then clear that $g = -C^{-1}\iota\,(1/\varphi_C)$ and $H = C^{-1} - \varphi_C gg'$. Since $Cg = -\iota\,(1/\varphi_C)$ and $CH = I + \iota g'$, it is not difficult to check that $C$, $H$ and $g$ relate as in (13). ☐

*Proof of Proposition 2 and of expressions (19) and (20)*:

Using (12), the first expression in footnote 8, and the fact that $\begin{pmatrix} -\Omega & w \\ w' & 1 \end{pmatrix} = \begin{pmatrix} -\hat{w}^{-1} & \iota \\ \iota' & 0 \end{pmatrix}$, the "bordered" direct Allais matrix can be related to the "bordered" indirect Allais matrix as follows:

$$(A9)\quad \begin{pmatrix} A - \frac{1}{\varphi_A}\,\iota\iota' & \iota \\ \iota' & 0 \end{pmatrix} = \begin{pmatrix} -\hat{w}^{-1} & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} C - \rho_C\,\iota\iota' & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} -\hat{w}^{-1} & \iota \\ \iota' & 0 \end{pmatrix},$$

which is expression (17) of the text.
Now let $D =_{\text{def}} (C - \rho_C\iota\iota')^{-1}$. Applying next the inversion formula for partitioned matrices yields:

$$(A10)\quad \begin{pmatrix} C - \rho_C\,\iota\iota' & \iota \\ \iota' & 0 \end{pmatrix}^{-1} = \begin{pmatrix} D - D\iota\,(\iota' D\iota)^{-1}\,\iota' D & D\iota\,(\iota' D\iota)^{-1} \\ (\iota' D\iota)^{-1}\,\iota' D & -(\iota' D\iota)^{-1} \end{pmatrix},$$

so that

$$(A11)\qquad A - \frac{1}{\varphi_A}\,\iota\iota' = \hat{w}^{-1}\left[D - \frac{1}{\iota' D\iota}\,(D + \hat{w})\,\iota\iota'\,(D + \hat{w})\right]\hat{w}^{-1}.$$

But using the Bartlett inverse formula,

$$(A12)\qquad \begin{aligned} D &= C^{-1} + \frac{\rho_C}{1 - \rho_C\varphi_C}\,C^{-1}\,\iota\iota' C^{-1} \\ &= \frac{1}{1 - \rho_C\varphi_C}\,[(1 - \rho_C\varphi_C)\,C^{-1} + \rho_C C^{-1}\,\iota\iota'\,C^{-1}], \end{aligned}$$

from which it follows that

$$(A13)\qquad D\iota = \frac{1}{1 - \rho_C\varphi_C}\,C^{-1}\iota, \quad \text{and} \quad \iota' D\iota = \frac{\varphi_C}{1 - \rho_C\varphi_C}.$$

Another way of writing (A11) is then

$$(A14)\quad \begin{aligned} &A - \frac{1}{\varphi_A}\,\iota\iota' \\ &\quad = \hat{w}^{-1}\left[D - \frac{(C^{-1}\iota + (1 - \rho_C\varphi_C)\,w)\,(\iota' C^{-1} + (1 - \rho_C\varphi_C)\,w')}{\varphi_C\,(1 - \rho_C\varphi_C)}\right]\hat{w}^{-1}. \end{aligned}$$

Putting the square bracket terms on the common denominator $\varphi_C\,(1 - \rho_C\varphi_C)$, substituting $D$ for the lower RHS of (A12), and rearranging then finally produces (19) in the text.

In a similar way, one can combine (13) and the second expression in footnote 8 to express the "bordered" indirect Allais matrix as a function of the "bordered" direct Allais matrix (since $\begin{pmatrix} \Omega & -w \\ w' & 1 \end{pmatrix} = \begin{pmatrix} \hat{w}^{-1} & \iota \\ -\iota' & 0 \end{pmatrix}$):

$$(A15)\quad \begin{pmatrix} C - \frac{1}{\varphi_C}\,\iota\iota' & \iota \\ \iota' & 0 \end{pmatrix} = \begin{pmatrix} \hat{w}^{-1} & -\iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} A - \rho_A\,\iota\iota' & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} \hat{w}^{-1} & \iota \\ -\iota' & 0 \end{pmatrix}.$$

This is expression (18) of the text. Exactly the same type of manipulations then result in expression (20). □

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