ANNALES D'ÉCONOMIE ET DE STATISTIQUE. – N° 43 — 1996

Monte Carlo Methodology for LM and LR Autocorrelation Tests in Multivariate Regression

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ABSTRACT. – The small sample critical values of three test statistics for vector autocorrelated errors are investigated in the context of several specific empirical models. Under the null hypothesis, two of the proposed statistics do not depend on nuisance parameters when the regressors are strongly exogenous, and their distributions are easy to estimate. We also propose a simple and accurate size correction for the Chi-square likelihood ratio test.

Méthodologie de simulation pour les tests LM et LR d'autocorrélation en régression multivariée

RÉSUMÉ. – Les valeurs critiques de trois statistiques de test d'autocorrélation multivariée sont étudiées, en petit échantillon, dans le contexte de plusieurs modèles empiriques spécifiques. Sous l'hypothèse nulle, deux des statistiques proposées ne dépendent pas de paramètres inconnus lorsque les régresseurs sont fortement exogènes, et leurs distributions sont faciles à estimer. Nous proposons également une correction simple et précise de la taille du test Chi-carré du rapport des vraisemblances.

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1 Introduction [1]

In important papers, GODFREY [1978a, 1978b] and BREUSCH and PAGAN [1980] show that the Lagrange multiplier (LM), or score, test statistic for autocorrelation in a single-equation linear model can be computed as TR^2, where T is the number of observations and where R^2 is the coefficient of determination in a regression of the ordinary least squares residuals on their lagged values and on the regressors of the model. GODFREY [1981] shows that this LM test also has power against moving average disturbances. The LM statistic can also be formulated in the multivariate regression case and is considerably easier to compute than its likelihood ratio (LR) counterpart, as emphasized in DESCHAMPS [1993].

In the single-equation case, the LM statistic converges in distribution to a monotonic function of a central F variate under the null hypothesis. In multivariate regressions, an LM test for vector autocorrelated errors happens to be identical to an LM test for the omission of lagged OLS residuals from an auxiliary reduced form equation. GODFREY [1988, p. 178] suggests using a likelihood ratio version of this test; we will call this version QLR for brevity. GUILKEY [1974] investigated the approximation of the critical values of a related statistic by those of the U distribution, described in ANDERSON [1958].

In Section 2 of this paper, we state forms of the LM and QLR statistics that do not depend on unknown parameters when the regressors are strongly exogenous and the null hypothesis is true. Indeed, the null distributions depend only on the regressor matrix; for an investigator studying models differing only by their dependent variables, such as the models in BARTEN [1989], this is a valuable feature, since one simulation can serve to estimate the exact size in all cases.

In Section 3, we briefly review some approximations of the LM, LR, and QLR null distributions. In the LM case, the F correction (valid only for single equation models) is stated. In the QLR case, the approximation is based on the U distribution. In the LR case, the proposed correction is based on the Bartlett adjustment factor (BARTLETT [1937]; LAWLEY [1956]), and only requires an accurate estimate of $E(LR)$, the expectation of the LR statistic.

In Section 4, we compare the effectiveness of the proposed approximations in the context of eight specifications based on two different models, involving one and two equations. In the first model, the regression equation includes

1. Previous versions of this paper were presented at the European Meeting of the Econometric Society, Maastricht, 1994; at the Center for Operations Research and Econometrics, University of Louvain, Belgium; at the Center for Economic Research, Tilburg University, The Netherlands; and at the Symposium in Honour of Professor A.P. Barten held in Leuven, Belgium, April 1995. I am indebted to A. Chesher and to anonymous referees for helpful comments. Any error is my own.

some dynamics, and the effect of initial conditions is investigated. The Bartlett correction of the LR statistic appears to perform well. However, an accurate naive estimation of $E(LR)$ would be impractical in cases involving more than two equations. In order to be able to generalize, we therefore propose the estimation of $E(LR)$ by a control variate method, and investigate the use of such control variate estimates in conjunction with the Bartlett correction. The results suggest that this procedure is satisfactory.

In Section 5, we apply the preceding techniques to more general multivariate regression models involving 4, 6, and 8 equations, with sample sizes of 28, 38, and 102 observations. Section 6 concludes.

2 The LM, LR and QLR Statistics

We consider the following regression equation:

$$Y = BX + U \tag{1}$$

$$U = \sum_{j=1}^{p} R_j U_{-j} + V \tag{2}$$

where Y is an $n \times T$ matrix of T observations on n dependent variables, B is an $n \times k$ matrix of coefficients, X is a $k \times T$ matrix of T observations on k regressors, U and V are $n \times T$ matrices of current disturbances with $\mathrm{vec}V \sim N(0, I_T \otimes \Sigma)$, Σ is a positive definite matrix of order n, the R_j are $n \times n$ matrices of autoregression coefficients, and the U_{-j} are $n \times T$ matrices of lagged disturbances. We let $R = (R_1 \dots R_p)$, and define:

$$Y_1 = \begin{pmatrix} Y_{-1} \\ \vdots \\ Y_{-p} \end{pmatrix} \tag{3}$$

$$X_1 = \begin{pmatrix} X_{-1} \\ \vdots \\ X_{-p} \end{pmatrix} \tag{4}$$

where the Y_{-i} and X_{-i} are $n \times T$ and $k \times T$ matrices of lagged observations on the dependent variables and on the regressors, respectively. The j-th intermediate lag can be omitted by skipping R_j, Y_{-j}, and X_{-j} in the definitions of R, Y_1, and X_1.

From DESCHAMPS [1993, Theorem 4, eq. 27], the LM statistic for the test of $H_0 : R = O$ against $H_1 : R \neq O$ can be computed as:

$$LM = T\,\mathrm{tr}\Big[\widehat{U}'(\widehat{U}\widehat{U}')^{-1}\widehat{U}\Big]\Big[\widehat{U}_1'(\widehat{U}_1 P_X \widehat{U}_1')^{-1}\widehat{U}_1\Big] \tag{5}$$

with the following definitions:

$$\widehat{U} = YP_X \tag{6}$$

$$P_X = I_T - X'(XX')^{-1}X \tag{7}$$

$$\widehat{U}_1 = Y_1 - \left[I_p \otimes YX'(XX')^{-1}\right]X_1 \tag{8}$$

where I_T and I_p are identity matrices of orders T and p. It follows from the same arguments as in WHITE [1987] that the statistic in equation (5) can be used when the regressors include lagged endogenous variables. It is not valid, however, when equation (1) is subject to cross-equation restrictions.

It is shown in Appendix A that LM in (5) is identical to the LM statistic for testing $R = O$ against $R \neq O$ in the auxiliary regression equation:

$$Y = BX + R\widehat{U}_1 + W \tag{9}$$

where $\widehat{U}_1$ is given by (8), and is treated as exogenous. Following GODFREY [1988, p. 178], we define the "quasi-likelihood ratio", or QLR, statistic from the LR analogue of this test:

$$QLR = T\Big(\ln\det T^{-1}\widehat{U}\widehat{U}' - \ln\det T^{-1}\widehat{W}\widehat{W}'\Big) \tag{10}$$

where $\widehat{U}$ is given by (6), and where $\widehat{W}$ is the matrix of OLS residuals from equation (9).

Finally, the true likelihood ratio test statistic may be written as:

$$LR = T(\ln\det\widehat{\Sigma}_0 - \ln\det\widehat{\Sigma}_1) \tag{11}$$

with:

$$\widehat{\Sigma}_0 = T^{-1}\widehat{U}\widehat{U}' = T^{-1}YP_XY' \tag{12}$$

$$\begin{aligned}\widehat{\Sigma}_1 = T^{-1}&[Y - \widehat{R}Y_1 - \widehat{B}X + \widehat{R}(I_p \otimes \widehat{B})X_1] \\ &\times [Y - \widehat{R}Y_1 - \widehat{B}X + \widehat{R}(I_p \otimes \widehat{B})X_1]'\end{aligned} \tag{13}$$

and where $\widehat{R}$ and $\widehat{B}$ are the maximum likelihood (ML) estimates of R and of B in the unconstrained model. These estimates can be obtained by the procedure described in DESCHAMPS [1993], and require extensive computation, especially when n is large. Note the theoretical inequality $LM \leq QLR \leq LR$, where $LM \leq QLR$ follows from BERNDT and SAVIN [1977], and where $QLR \leq LR$ follows from the fact that the ML estimates $\widehat{R}$ and $\widehat{B}$ minimize the generalized variance $\det\widehat{\Sigma}_1$.

In Appendix B, we show that under the null hypothesis, LM and QLR can be written as functions involving only X and an $n \times (T + p)$ matrix $\mathcal{E}$ of independent standard normal variables, so that they are independent of (B, R, Σ).

Indeed, let E_{-i}, for $i = 1, \ldots, p$, be the $n \times T$ submatrix consisting of those columns in $\mathcal{E}$ with indices ranging from $p+1-i$ to $T+p-i$; let E consist of the last T columns of $\mathcal{E}$; and let:

$$E_1 = \begin{pmatrix} E_{-1} \\ \vdots \\ E_{-p} \end{pmatrix}. \tag{14}$$

If H_0 is true, the LM statistic in equation (5) is identical to:

$$LM_0 = T \operatorname{tr} A_{11}^{-1} A_{12} A_{22}^{-1} A_{12}' \tag{15}$$

where:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}' & A_{22} \end{pmatrix} = \begin{pmatrix} E \\ E_* \end{pmatrix} P_X (E' \quad E_*') \tag{16}$$

$$E_* = E_1 - [I_p \otimes EX'(XX')^{-1}]X_1. \tag{17}$$

Similarly, if H_0 is true, the QLR statistic in equation (10) is identical to:

$$QLR_0 = T(\ln \det EP_X E' - \ln \det EP_X Q_1 P_X E') \tag{18}$$

where:

$$Q_1 = I_T - E_*'(E_* P_X E_*')^{-1} E_* \tag{19}$$

and P_X, E_* are given by (7) and (17).

3 Asymptotic Approximations

In the single-equation case ($n = 1$), an "F form of the LM statistic" can be motivated by the omitted variables interpretation of the LM test [Harvey (1990), p. 174], and defined as:

$$LMF = \frac{T-k-p}{p} \left(\frac{LM}{T-LM} \right). \tag{20}$$

Under the null hypothesis, LMF follows *asymptotically* an $F_{p,T-k-p}$ distribution. Asymptotic critical values for the LM test are obtained from equation (20) as:

$$LM(F_\alpha) = \frac{T}{\left(1 + \dfrac{T-k-p}{pF_\alpha}\right)} \tag{21}$$

where F_α is such that $P(F_{p,T-k-p} \leq F_\alpha) = 1 - \alpha$.

As is well known, when $n > 1$, there is no exact transformation of the LM statistic for omitted variables that follows an F distribution [2]. However, the U distribution, which is extensively described in ANDERSON [(1958), pp. 191-210], would give the exact small sample null distribution of QLR in (10) under the strong exogeneity of $\widehat{U}_1$ in (9). Since $e^{-QLR/T}$ converges in distribution to $U_{n,np,T-k-np}$, we may define an asymptotic null distribution of QLR as:

$$F(QLR) = 1 - P_U(e^{-QLR/T}) \tag{22}$$

where $P_U(.)$ is the cumulative distribution function of $U_{n,np,T-k-np}$. $P_U(.)$ is given explicitly by ANDERSON [1958] in some special cases and can also be approximated by a very accurate asymptotic expansion. Upon inverting (22), we obtain the following asymptotic critical values:

$$QLR(U_\alpha) = -T \ln U_\alpha \tag{23}$$

where U_α is such that $P_U(U_\alpha) = \alpha$ (note that the transformation from QLR to U is decreasing).

Finally, LAWLEY [1956] shows that the corrected LR statistic $LR \times [d/E(LR)]$ has, under the null hypothesis and ignoring quantities of order T^{-2}, the same moments as the asymptotic central Chi-square; here $E(LR)$ is the expected value of LR under the null hypothesis and d denotes the number of degrees of freedom of the asymptotic distribution [3]. Even though $E(LR)$ is unknown, it can be estimated by Monte Carlo methods. Assuming that such an estimate $\widehat{E}(LR)$ is available, we may define an asymptotic LR critical value as:

$$LR(\chi^2_\alpha) = \frac{\widehat{E}(LR)}{n^2 p} \chi^2_\alpha \tag{24}$$

where χ^2_α is such that $P(\chi^2_{n^2p} \leq \chi^2_\alpha) = 1 - \alpha$.

4 Some Monte Carlo Results

The results in Section 2 imply that the small sample distributions of LM_0 and QLR_0 in (15) and (18) depend only on the matrix X when the latter is

2. MUIRHEAD [1982, p. 479] gives an asymptotic expansion that can be applied for an LM test of omitted variables with strongly exogenous regressors. This approximation was investigated and found to be unsatisfactory in the present case.
3. This correction is originally due to BARTLETT [1937]; for a recent treatment, see BARNDORFF-NIELSEN and COX [1984]. The Edgeworth correction of the LR statistic proposed by ROTHENBERG [1984] in the context of linear GLS models with linear constraints can also be shown to be similar to the Bartlett correction.

strongly exogenous. Hence, in this case, the *true* null distributions can be estimated. By contrast, simulating the null distribution of LR in equation (11) requires replications of $Y = BX + V$ and of $Y_1 = (I_p \otimes B)X_1 + V_1$, with $V = \Sigma^{1/2}E$ and with $V_1 = (I_p \otimes \Sigma^{1/2})E_1$; since Y and Y_1 depend on the unknown B and Σ, the obtained empirical distribution is conditional on estimates of these matrices, say $\tilde{B}$ and $\tilde{\Sigma}$ (this is the parametric bootstrap). If $\tilde{B}$ and $\tilde{\Sigma}$ are poor estimates of B and Σ, the conditional bootstrap distribution may well be a poor approximation of the true small sample distribution, unless, of course, sensitivity to $\tilde{B}$ and $\tilde{\Sigma}$ is small.

When the regression equation is dynamic, the simulated distributions of LM_0, QLR_0, and LR depend on $\tilde{B}$, $\tilde{\Sigma}$, and on the initial values of the regressors, since X must be simulated recursively. However, equations (15) and (18) remain valid.

In this section, we will first investigate, in some specific cases, the sensitivity of the simulated null distributions to $\tilde{B}$, to $\tilde{\Sigma}$, and to the initial values of the regressors; we will then compare the performance of the approximations of Section 3 in those cases.

The chosen specifications are the first eight of the 12 summarized in Table 1, and described more extensively in Appendix C. The last four specifications will be studied in Section 5.

TABLE 1

Specification Characteristics [4].

Specification	n	p	T	k	s_{1952}	$\tilde{B}$, $\tilde{\Sigma}$	Model
1	1	1	28	4	0.03	OLS	Pesaran-Evans
2	1	1	14	4	0.03	OLS	Pesaran-Evans
3	1	1	28	4	0.25	OLS	Pesaran-Evans
4	1	1	14	4	0.25	OLS	Pesaran-Evans
5	2	1	42	4		OLS	Berndt-Savin
6	2	1	21	4		OLS	Berndt-Savin
7	2	1	42	4		ML	Berndt-Savin
8	2	1	21	4		ML	Berndt-Savin
9	4	1	38	7		ML	Annual Rotterdam
10	4	1	28	7		ML	Annual Rotterdam
11	6	1	38	9		ML	Annual Rotterdam
12	8	2	102	10		ML	Quarterly Rotterdam

Specifications 1 to 4 are based on the preferred savings equation of PESARAN and EVANS [1984], where a quasi-differenced savings ratio $s_t - \theta_t s_{t-1}$ is explained in particular by the sum of the lagged savings ratio and of the lagged capital gains ratio. This is a dynamic single-equation model with four explanatory variables. The data for this model are annual, and are

4. n is the number of equations; p is the number of autoregressive lags; T is the number of observations; k is the number of regressors per equation; s_{1952} is the initial savings ratio; ML denotes maximum likelihood estimation; OLS denotes ordinary least squares estimation.

distributed as part of the MICROFIT econometric package. Specifications 1 and 3 use the full sample period, 1953 to 1981. Specifications 2 and 4 differ from 1 and 3 in that only half the sample observations were used (the sample period is in this case 1953 to 1967).

The residuals in the Pesaran-Evans model show practically no autocorrelation, and the OLS and ML estimates of B and Σ are almost identical. Hence, the OLS estimates were the only ones used. However, since the regression equation in this model is dynamic, the sensitivity of the estimated null distributions to initial conditions must be investigated. For this reason, two different values of the initial savings ratio (s_{1952}) were used in the recursive simulation of X. The first one (in specifications 1 and 2) is the observed value of $s_{1952} = 0.03$. The second one (in specifications 3 and 4) was chosen as 0.25, this number being a plausible upper bound.

Specifications 5 to 8 are based on the three-input factor shares system of BERNDT and SAVIN [1975]. This model is static, involves two equations, and four regressors per equation; it was estimated on the basis of annual data, covering the period 1929 to 1971, and published in the Appendix of the Berndt-Savin article.

In the case of the Berndt-Savin model, the residuals exhibit severe autocorrelation, with an LM statistic of 35.75 (the 5% Monte Carlo critical value being 11.67). The ML and OLS estimates of B and Σ differ substantially, and the OLS estimate of Σ is probably inconsistent. Consequently, this model gives us an opportunity to investigate the sensitivity of the estimated null distribution of LR to the nuisance parameters B and Σ. For this reason, the OLS estimates of B and Σ were used in specifications 5 and 6, whereas specifications 7 and 8 are based on the ML estimates.

Again, specifications 6 and 8 differ from specifications 5 and 7 in that only half the sample observations were used.

It is hoped that this (admittedly rather limited) "experimental design", together with the four remaining specifications to be studied in Section 5, covers an adequate range of situations. An alternative methodology, used in, e.g., HENDRY and HARRISON [1974] and KIVIET [1986], models the regressors as stochastic processes and includes the parameters of these processes in the experimental design. For realistic multivariate regression models, however, where n typically exceeds five, the dimension of the parameter space becomes so large that this procedure becomes impractical.

The sensitivity of the three null distributions with respect to the choice of s_{1952}, $\tilde{B}$, and $\tilde{\Sigma}$ was first investigated by estimating the first two moments of each statistic in the context of specifications 1 to 8. Table 2 presents such estimates, each based on 20000 replications of the statistics when $R = O$. It is apparent that the moments for those specifications that involve the same sample size do not differ substantially from each other, except in one case, that of the LR statistic with $n = 1$ and $T = 14$ (specifications 2 and 4). A rigorous test of the equality of expectations and variances confirms this observation: at the 5% level, 1.66 differs significantly from 1.82 in the third column of Table 2, and 5.15 differs significantly from 5.63 in the fourth column. However, it may be checked that in specifications with the same sample size, all the other differences are insignificant at the 5% level. So,

TABLE 2

Estimated Moments.

Specification	$E(LM)$	$V(LM)$	$E(LR)$	$V(LR)$	$E(QLR)$	$V(QLR)$
1	1.16	2.38	1.30	3.28	1.25	3.05
2	1.23	2.18	1.66	5.15	1.40	3.54
3	1.16	2.33	1.30	3.29	1.25	3.06
4	1.22	2.13	1.82	5.63	1.41	3.55
5	4.80	9.68	5.27	13.47	5.16	13.04
6	5.66	10.36	6.96	21.22	6.74	19.89
7	4.80	9.68	5.22	13.15	5.16	13.04
8	5.66	10.36	6.96	21.47	6.74	19.89

with the exception of the LR test in the half-sample Pesaran-Evans model, any difference in the conditional moments can be expected to be less than experimental error, even with the large number of replications used. Similar conclusions are reached upon comparing the critical values, as will be seen from the first, third, and sixth columns of Table 3.

TABLE 3

Ratios of Estimated to Asymptotic Critical Values.

Specification	α	$\frac{LM_\alpha}{\chi^2_\alpha}$	$\frac{LM_\alpha}{LM(F_\alpha)}$	$\frac{LR_\alpha}{\chi^2_\alpha}$	$\frac{LR_\alpha}{LR(\chi^2_\alpha)}$	$\frac{QLR_\alpha}{QLR(U_\alpha)}$	$\frac{QLR_\alpha}{\chi^2_\alpha}$
	0.10	1.15	0.99	1.29	0.99	1.01	1.26
1	0.05	1.15	1.01	1.32	1.01	1.01	1.26
	0.01	1.07	0.99	1.26	0.97	0.96	1.19
	0.10	1.21	0.86	1.63	0.99	0.84	1.38
2	0.05	1.13	0.86	1.62	0.98	0.82	1.35
	0.01	0.98	0.86	1.58	0.95	0.80	1.32
	0.10	1.18	1.01	1.32	1.02	1.01	1.26
3	0.05	1.13	0.99	1.30	0.99	0.99	1.24
	0.01	1.05	0.97	1.25	0.96	1.01	1.25
	0.10	1.20	0.86	1.78	0.98	0.86	1.40
4	0.05	1.13	0.86	1.73	0.95	0.83	1.37
	0.01	0.96	0.84	1.64	0.90	0.79	1.29
	0.10	1.16		1.30	0.99	1.09	1.29
5	0.05	1.13		1.30	0.99	1.08	1.28
	0.01	1.08		1.30	0.99	1.07	1.27
	0.10	1.29		1.70	0.97	1.13	1.64
6	0.05	1.23		1.66	0.95	1.11	1.62
	0.01	1.12		1.65	0.95	1.08	1.56
	0.10	1.16		1.29	0.99	1.09	1.29
7	0.05	1.13		1.29	0.99	1.08	1.28
	0.01	1.08		1.29	0.99	1.07	1.27
	0.10	1.29		1.68	0.97	1.13	1.64
8	0.05	1.23		1.68	0.96	1.11	1.62
	0.01	1.12		1.64	0.94	1.08	1.56

We now turn to a preliminary investigation of the quality of the asymptotic approximations described in Section 3. Table 3 is based on the same 20000 replications as Table 2, and presents the ratios of empirical to asymptotic critical values for the three statistics. The asymptotic critical values considered were the Chi-square quantiles (χ^2_α) and the ones discussed in Section 3 ($LM(F_\alpha)$, $LR(\chi^2_\alpha)$ based on the estimates of $E(LR)$ given in Table 2, and $QLR(U_\alpha)$). The systematic order of the entries in the first, third, and sixth columns of Table 3 is due to the previously mentioned inequalities $LM \leq QLR \leq LR$.

In Table 3, the Chi-square critical values grossly underestimate the true LR ones (with differences ranging from 25% to 78% in the third column), and markedly, though less seriously, misestimate the LM ones (with differences ranging from -4% to 29% in the first column). Among the small sample methods of Section 3, the F approximation of the LM distribution, when defined, performs better than the U approximation of the QLR distribution, as can be seen by comparing the entries in the second and fifth columns; and the performance of the Bartlett approximation is consistently the best, as can be seen from the fourth column.

A computation of the differences between estimated and nominal sizes for the same combinations of statistics and asymptotic approximations as in Table 3 revealed essentially the same patterns; the results of this computation are therefore not reported.

For large values of n, however, simulating the distribution of LR becomes impractical. For instance, estimating the homogeneous quarterly Rotterdam model in DESCHAMPS [1993], where $n = 8$, $T = 102$, $k = 10$, and $p = 2$, requires the iterated computation of $(Z'\widehat{\Omega}^{-1}Z)^{-1}Z'\widehat{\Omega}^{-1}y$, where Z is 816×80 and where $\widehat{\Omega}$ is symmetric of order 816.

Since LR and LM are likely to be highly correlated and since the null distribution of LM can be estimated relatively inexpensively and (conditionally on $\tilde{B}$, $\tilde{\Sigma}$, and s_{1952} in specifications 1 to 4) to an arbitrary degree of precision, the solution that immediately comes to mind is the use of LM as a control variate for LR. Control variate procedures have been proposed for estimating moments, tail areas, or quantiles. In the first context, a very attractive procedure, recently proposed by DAVIDSON and MACKINNON [1992], only involves regressing the simulated LR values on the difference between the simulated LM values and their expectation. We call this method CVE. When estimating tail areas, Davidson and MacKinnon also propose a regression, this time of a dependent indicator variable on an independent one, which is adjusted in such a way that it has zero expectation. Davidson and MacKinnon call this method BCV1. The method proposed by these authors for quantile estimation does not appear to be easily generalizable to the present, non normal, case. The simulation evidence presented by Davidson and MacKinnon illustrates that the potential efficiency gains are much larger in the first context (CVE) than in the second (BCV1).

In Table 4, we present the results of the CVE estimation of $E(LR)$, as well as the sample means, based on 100 replications of specifications 1 to 8. The "exact" expectations of the null LM distributions, which are required by the control variate method, were estimated using independent simulation runs of 100000 replications. The last row of Table 4 reports the coefficients of

TABLE 4

Control Variate (CVE) and Naive Estimation of $E(LR)$ (Standard Errors in Parentheses).

Specification	1	2	3	4	5	6	7	8
CVE	1.33 (0.03)	1.62 (0.06)	1.32 (0.04)	1.93 (0.14)	5.26 (0.03)	6.81 (0.07)	5.24 (0.03)	6.78 (0.06)
Naive	1.12 (0.16)	1.43 (0.21)	1.48 (0.22)	1.69 (0.24)	5.94 (0.43)	6.47 (0.45)	5.80 (0.38)	6.39 (0.39)
R^2	0.98	0.93	0.97	0.66	0.99	0.97	0.99	0.98

determination of the regressions of LR on $LM - E(LM)$. The efficiency gain of the control variate over the naive method is $(1 - R^2)^{-1}$. This efficiency gain is indeed substantial even for small values of T. In the worst-performing case (specification 4), three times as many replications would be required for a naive estimation of $E(LR)$ to yield the same efficiency as the CVE method. In view of the excellent performance of the Bartlett correction illustrated in Table 3, the usefulness of applying equation (24) with $E(LR)$ estimated by CVE in a small simulation run clearly deserves investigation.

This investigation was done in the following way. We computed 200 estimates of the last quantiles of the null LR distributions in specifications 1 to 8, using both the Bartlett approximation (with $E(LR)$ estimated by CVE) and the naive method that uses the samples quantiles. Each of the 200 estimates was based on 100 different observations in the full data set. We then estimated the biases and root mean squared errors for each method, using the population quantiles from the full simulations as the true values. KENDALL and STUART [1977, p. 251] show that the bias of the sample quantiles is $O(N^{-1})$.

The results of these experiments are reported in Table 5, where the estimated relative bias is defined as:

$$ERB = \frac{\overline{LR}_\alpha - LR_\alpha}{LR_\alpha}, \tag{25}$$

where LR_α is the LR population quantile computed from the full simulation run, and where:

$$\overline{LR}_\alpha = \frac{1}{200}\sum_{i=1}^{200} LR_\alpha^i$$

$$LR_\alpha^i = \begin{cases} \dfrac{\widehat{E}_i}{n^2 p}\chi^2_\alpha & \text{(for the Bartlett approximation)} \\ \text{the sample quantile} & \text{(for the naive estimation)} \end{cases}$$

$\widehat{E}_i$ being the i-th CVE estimation of $E(LR)$, and χ^2_α being a percentage point of the central Chi-square with n^2p degrees of freedom. The root mean

TABLE 5

Bartlett Approximation Versus Naive Estimation of the LR Critical Values [5].

Specification	α	Bartlett		Naive	
		ERB	RMSE	ERB	RMSE
	0.10	0.01	0.01	-0.03	0.04
1	0.05	-0.01	0.01	-0.04	0.07
	0.01	0.03	0.02	-0.08	0.14
	0.10	0.01	0.02	-0.02	0.05
2	0.05	0.02	0.03	-0.04	0.08
	0.01	0.05	0.05	-0.12	0.18
	0.10	-0.01	0.01	-0.02	0.04
3	0.05	0.02	0.01	-0.04	0.06
	0.01	0.05	0.03	-0.09	0.14
	0.10	0.03	0.02	-0.02	0.05
4	0.05	0.06	0.04	-0.04	0.08
	0.01	0.12	0.10	-0.11	0.16
	0.10	0.01	0.01	-0.02	0.06
5	0.05	0.01	0.01	-0.03	0.08
	0.01	0.01	0.01	-0.08	0.18
	0.10	0.02	0.02	-0.02	0.08
6	0.05	0.04	0.05	-0.02	0.12
	0.01	0.05	0.08	-0.08	0.23
	0.10	0.02	0.01	-0.02	0.07
7	0.05	0.02	0.02	-0.02	0.10
	0.01	0.02	0.02	-0.07	0.20
	0.10	0.02	0.03	-0.01	0.08
8	0.05	0.03	0.03	-0.02	0.11
	0.01	0.05	0.08	-0.06	0.24

squared errors (of $\overline{LR}_\alpha$) are estimated as:

$$RMSE = \frac{1}{\sqrt{200}} \sqrt{\frac{1}{200} \sum_{i=1}^{200} (LR_\alpha^i - LR_\alpha)^2}. \tag{26}$$

The results in Table 5 are indeed encouraging. The absolute value of the estimated relative bias is generally smaller for the Bartlett approximation. Furthermore, the root mean squared error estimates are uniformly reduced, sometimes by an order of magnitude (for specifications 5 and 7, corresponding to the largest values of T). We therefore feel confident that the previous methodology can be validly applied to specifications 9, 10, 11, and 12, for which an accurate naive estimation of the LR quantiles is unfeasible. This is done in the next section.

5. ERB denotes the estimated relative bias (equation (25)) and RMSE denotes the estimated root mean squared error (equation (26)).

5 More Monte Carlo Results

Specifications 9, 10, 11 and 12 are based on the Rotterdam system of demand equations, Barten [1969], which is a static demand model. For the three first specifications (9, 10, and 11), we used annual British data, published in the *Economic Trends Annual Supplement* [1993], on the expenditures for Food, Drink and Tobacco, Clothing and Footwear, Energy Products, Other Goods, Housing Services, and Other Services for the period 1952 to 1991. In specification 9, the last two groups of commodities (Housing Services and Other Services) were omitted, leaving $n = 4$ equations and $T = 38$ observations after differencing. Specification 10 is based on the same model as specification 9, but for the restricted sample period 1952 to 1981 (the LR statistic would not be defined for $T = 38/2 = 19$). Specification 11 involves all seven commodities and the full sample period. Specification 12 is the quarterly homogeneous Rotterdam model described in Deschamps [1993]. Additional details are given in Appendix C.

Table 6 reports the estimates of the first moments of LM, QLR, and LR in specifications 9 to 12. In each case, $E(LM)$ is estimated from 100000 replications; $E(QLR)$ is estimated from 20000 replications; and $E(LR)$ is estimated by the CVE method of the previous section, with the required precise estimate of $E(LM)$ given by the first column of the table. As before, R^2 is the coefficient of determination in the control variate regression, and N is the number of replications on which the CVE estimates are based.

Table 6

Estimated Moments (Standard Errors in Parentheses).

Specification	$E(LM)$	$E(QLR)$	$E(LR)$	R^2	N
9	20.38 (0.02)	24.07 (0.06)	28.92 (0.20)	0.87	300
10	22.23 (0.02)	28.81 (0.07)	38.09 (0.23)	0.68	1000
11	48.84 (0.03)	64.16 (0.10)	87.29 (0.39)	0.67	850
12	145.03 (0.05)	169.08 (0.15)	180.56 (0.49)	0.95	100

The figures in Table 7 are based on 20000 replications of the LM and QLR statistics, and on the estimates of $E(LR)$ given in Table 6. An examination of Table 7 confirms that the LM critical values are the closest to the asymptotic Chi-square ones, and that the asymptotic critical values usually underestimate the small sample ones (by amounts varying from 8% to 28% in the LM case, and from 41% to 142% in the LR case!) It should be noted, however, that in the second column of Table 7, the approximate critical value of $LR(\chi^2_\alpha)$ is viewed as if it were the true small sample one.

TABLE 7

Ratios of Estimated to Asymptotic Critical Values.

Specification	α	$\frac{LM_\alpha}{\chi^2_\alpha}$	$\frac{LR(\chi^2_\alpha)}{\chi^2_\alpha}$	$\frac{QLR_\alpha}{QLR(U_\alpha)}$	$\frac{QLR_\alpha}{\chi^2_\alpha}$
	0.10	1.20	1.81	1.04	1.50
9	0.05	1.18	1.81	1.04	1.49
	0.01	1.13	1.81	1.02	1.47
	0.10	1.28	2.38	1.04	1.78
10	0.05	1.25	2.38	1.04	1.79
	0.01	1.17	2.38	1.03	1.77
	0.10	1.27	2.42	1.04	1.77
11	0.05	1.25	2.42	1.03	1.77
	0.01	1.20	2.42	1.04	1.78
	0.10	1.11	1.41	1.02	1.32
12	0.05	1.10	1.41	1.02	1.32
	0.01	1.08	1.41	1.02	1.32

In the third column of Table 7, the U approximation of the QLR critical values appears to be reasonably good, and reasonably independent of the sample size T, with errors ranging from 2% to 4%.

We also computed the differences between estimated and nominal sizes; they ranged from 0.03 to 0.46 for the χ^2-based LM test, from 0.33 to 0.93 for the χ^2-based LR test, and from 0.01 to 0.04 for the U-based QLR test.

It should be noted that even in the largest model (specification 12), 10000 replications of LM_0 in (15) require only about one hour on a computer equipped with an Intel 80486DX-33 processor, using GAUSS version 3.1.5; and 10000 replications of QLR_0 in (18) require about two hours. Since the Monte Carlo estimates of the LM critical values can be considered exact (apart from experimental error) when the regression equation is static, some may prefer to use them all the time in such a case. On the other hand, even 100 replications of LR in (11) can take several days of computer time.

6 Conclusions

This paper has proposed methodologies for estimating the small sample sizes of three known tests for autocorrelated errors in multivariate regression models. The first one, for the LM test, is based on pure simulation; the second, for the QLR test, is based on an asymptotic distribution; and the third, for the LR test, is based on an estimated correction. When the regressors are strongly exogenous, the first method can be considered an exact one apart from experimental error; it is also quite inexpensive. The second method does not require simulation but is only moderately accurate in small samples. On the other hand, the U-based QLR test appears to perform better than what the evidence presented by GUILKEY [1974] suggests. Part of the reason may be that Guilkey's statistic is not equal to QLR unless $\widehat{U}_1 P_X = \widehat{U}_1$. The third method is the costliest, but apparently gives excellent results. It would appear to be most useful in the cases where the LR test can be expected to be more powerful than either the LM or QLR tests. Whereas our version of the LR test (equation 11) is based on the likelihood function conditional on the first observations, a version of this test based on the unconditional likelihood can also be formulated; see BEACH and MACKINNON [1979]. In this case, when $R \neq O$, the unconditional likelihood function can give considerable weight to the first observations, so that a test based on this function can be expected to be powerful. On the other hand, if $R = O$, the difference between the two LR statistics is $O(T^{-1})$, so that the Bartlett correction can be expected to remain applicable. However, the unconditional likelihood is considerably more difficult to maximize than the conditional one.

When the regression equation is dynamic, limited evidence suggests that the critical values of LM and QLR are less sensitive than those of LR to nuisance parameters and initial conditions. When the regression equation is static, the critical values of LR appear to be relatively insensitive to nuisance parameters (those of LM and QLR are invariant).

The three statistics obey the theoretical inequality $LM \leq QLR \leq LR$, and the estimated small sample critical values of LM are usually greater than the corresponding Chi-square ones. This implies that the Chi-square approximation of the LM critical values is usually better than in the two other cases.

Additional, and extensive, Monte Carlo evidence would have to be obtained before the results in this paper could be extended to models including lagged endogenous variables. Even in a model involving only two equations, however, the parameter space has such high dimension that a full Monte Carlo investigation is a major undertaking.

For large values of n, the χ^2-based versions of the LM, QLR, and LR tests should be used with caution, especially in the LR and QLR cases. Similarly, some heuristic small sample corrections, such as multiplying both the LM and LR statistics by $(T-k)/T$ and using the Chi-square critical values, are inappropriate in the present context. Indeed, when n and k are large and T is small, the variance to mean ratio of the small sample LM distribution can become considerably less than the value of $2T/(T-k)$, which would be applicable if this procedure were correct.

APPENDIX A

An Equivalent Form of the LM Test

In this Appendix, we show that LM in (5) is identical to the LM statistic for testing $R = O$ in (9). Since $\widehat{W}P_X = \widehat{W}$ and since $YP_X = \widehat{U}$, the estimated form of (9) can be multiplied by P_X to yield:

$$\widehat{U} = \widehat{R}\widehat{U}_1 P_X + \widehat{W} \tag{27}$$

with:

$$\begin{aligned} \widehat{R} &= \widehat{U}P_X\widehat{U}_1'(\widehat{U}_1P_X\widehat{U}_1')^{-1} \\ &= \widehat{U}\widehat{U}_1'(\widehat{U}_1P_X\widehat{U}_1')^{-1}. \end{aligned} \tag{28}$$

If we denote the estimated contemporaneous covariance matrix of W by $\widehat{\Omega}_0$ when $R = O$, and by $\widehat{\Omega}_1$ in the unconstrained case, we have:

$$\widehat{\Omega}_0 = T^{-1}\widehat{U}\widehat{U}' \tag{29}$$

$$\begin{aligned} \widehat{\Omega}_1 &= T^{-1}(\widehat{U} - \widehat{R}\widehat{U}_1P_X)(\widehat{U} - \widehat{R}\widehat{U}_1P_X)' \\ &= T^{-1}(\widehat{U}\widehat{U}' - \widehat{R}\widehat{U}_1\widehat{U}' - \widehat{U}\widehat{U}_1'\widehat{R}' + \widehat{R}\widehat{U}_1P_X\widehat{U}_1'\widehat{R}') \\ &= T^{-1}\Big[\widehat{U}\widehat{U}' - \widehat{U}\widehat{U}_1'(\widehat{U}_1P_X\widehat{U}_1')^{-1}\widehat{U}_1\widehat{U}'\Big] \end{aligned} \tag{30}$$

where we have used (28), and $\widehat{R}\widehat{U}_1\widehat{U}' = \widehat{R}\widehat{U}_1P_X\widehat{U}_1'\widehat{R}'$. From Berndt and Savin [1977], the LM statistic for testing $R = O$ in (9) can be written as $\widetilde{LM} = T\mathrm{tr}\widehat{\Omega}_0^{-1}(\widehat{\Omega}_0 - \widehat{\Omega}_1)$. Since, from (29) and (30):

$$\widehat{\Omega}_0 - \widehat{\Omega}_1 = T^{-1}\widehat{U}\widehat{U}_1'(\widehat{U}_1P_X\widehat{U}_1')^{-1}\widehat{U}_1\widehat{U}' \tag{31}$$

$$\widehat{\Omega}_0^{-1} = T(\widehat{U}\widehat{U}')^{-1} \tag{32}$$

it is seen that:

$$\begin{aligned} \widetilde{LM} &= T\mathrm{tr}T(\widehat{U}\widehat{U}')^{-1}T^{-1}\widehat{U}\widehat{U}_1'(\widehat{U}_1P_X\widehat{U}_1')^{-1}\widehat{U}_1\widehat{U}' \\ &= T\mathrm{tr}\Big[\widehat{U}'(\widehat{U}\widehat{U}')^{-1}\widehat{U}\Big]\Big[\widehat{U}_1'(\widehat{U}_1P_X\widehat{U}_1')^{-1}\widehat{U}_1\Big] \\ &= LM. \end{aligned}$$

APPENDIX B

Invariance of the LM and QLR Statistics

In this Appendix, we prove that under the null hypothesis that $R = O$, LM in (5) and LM_0 in (15) are identical, and that QLR in (10) is identical to QLR_0 in (18).

In view of (1), (2), and (6), we have under H_0:

$$\widehat{U} = YP_X = (BX + V)P_X = VP_X. \tag{33}$$

Furthermore, (8) implies:

$$\begin{aligned} \widehat{U}_1 &= (I_p \otimes B)X_1 + V_1 - [I_p \otimes (B + VX'(XX')^{-1})]X_1 \\ &= V_1 - \left[I_p \otimes VX'(XX')^{-1}\right]X_1 \end{aligned} \tag{34}$$

where V_1 is defined, similarly to E_1 in (14), by lagging the elements of an $n \times (T+p)$ matrix $\mathcal{V}$ with $\text{vec}\mathcal{V} \sim N(0, I_{T+p} \otimes \Sigma)$ and where V consists of the last T columns of $\mathcal{V}$.

If we now let $E = \Sigma^{-1/2}V$ and $E_1 = (I_p \otimes \Sigma^{-1/2})V_1$, it is clear that $\text{vec}E \sim N(0, I_{nT})$ and that E_1 satisfies (14). From (33) and (34), we then have:

$$\Sigma^{-1/2}\widehat{U} = EP_X \tag{35}$$

$$(I_p \otimes \Sigma^{-1/2})\widehat{U}_1 = E_1 - [I_p \otimes EX'(XX')^{-1}]X_1 = E_*. \tag{36}$$

Equation (5) can now be written as:

$$\begin{aligned} LM &= T\text{tr}[\widehat{U}'\Sigma^{-1/2}(\Sigma^{-1/2}\widehat{U}\widehat{U}'\Sigma^{-1/2})^{-1}\Sigma^{-1/2}\widehat{U}] \\ &\quad \times [\widehat{U}_1'(I_p \otimes \Sigma^{-1/2})((I_p \otimes \Sigma^{-1/2})\widehat{U}_1 P_X \widehat{U}_1'(I_p \otimes \Sigma^{-1/2}))^{-1} \\ &\qquad \times (I_p \otimes \Sigma^{-1/2})\widehat{U}_1] \\ &= T\text{tr}\left[P_X E'(EP_X E')^{-1}EP_X\right]\left[E_*'(E_* P_X E_*')^{-1}E_*\right] \\ &= T\text{tr}(EP_X E')^{-1}(EP_X E_*')(E_* P_X E_*')^{-1}(E_* P_X E') \\ &= T\text{tr}A_{11}^{-1}A_{12}A_{22}^{-1}A_{12}' \end{aligned}$$

which proves the first of our claims.

We now show that $QLR = QLR_0$. From (29) and (35), we see that:

$$\ln\det\widehat{\Omega}_0 = \ln\det T^{-1}EP_X E' + \ln\det\Sigma,$$

and, if we let:

$$\widehat{Q}_1 = I_T - \widehat{U}_1'(\widehat{U}_1 P_X \widehat{U}_1')^{-1}\widehat{U}_1$$

we also have, from (30) and (35):

$$\begin{aligned}\ln\det\widehat{\Omega}_1 &= \ln\det T^{-1}\widehat{U}\widehat{Q}_1\widehat{U}' \\ &= \ln\det T^{-1}EP_X\widehat{Q}_1P_XE' + \ln\det\Sigma.\end{aligned}$$

Using (36) and the definition of Q_1 in (19), we can show that:

$$\widehat{Q}_1 = I_T - E_*'(E_*P_XE_*')^{-1}E_* = Q_1.$$

QLR in (10) can now be written as:

$$\begin{aligned}QLR &= T(\ln\det\widehat{\Omega}_0 - \ln\det\widehat{\Omega}_1) \\ &= T(\ln\det T^{-1}EP_XE' + \ln\det\Sigma \\ &\qquad - \ln\det T^{-1}EP_XQ_1P_XE' - \ln\det\Sigma) = QLR_0,\end{aligned}$$

which proves our second claim.

APPENDIX C

The Chosen Specifications

In this Appendix, we give a brief description of the models on which specifications 1 to 12 are based. Specifications 1 to 4 are based on the "preferred savings equation" in PESARAN and EVANS [1984], which is equation (12) in that article. Following the definitions of these authors, we denote by s_t the ratio of real savings to real income; by θ_t the ratio of nominal income at period $t-1$ to nominal income at period t; by g_t the ratio of nominal capital gains to nominal income; by z_t the ratio of the nominal value of shares to nominal income; by π_t the rate of inflation; and by y_t real personal disposable income. For $t = 1, \ldots, T$, the Pesaran-Evans savings equation can then be written as:

$$\text{(37)} \qquad s_t - \theta_t s_{t-1} = \alpha(1-\theta_t) + \beta\left[\frac{\pi_t}{(1+\pi_t)y_t}\right] + \gamma\theta_t(s_{t-1} + g_{t-1}) + \delta(z_t - \theta_t z_{t-1}) + u_t$$

where α, β, γ, and δ are regression coefficients and u_t is an error term.

Specifications 5 to 8 are based on the translog factor shares model described in BERNDT and SAVIN [1975, equation 37]. This model can be written as:

$$\text{(38)} \quad w_{it} = \alpha_i + \sum_{j=1}^{3} \gamma_{ij} \ln x_{jt} + u_{it} \quad (\text{for } t = 1, \ldots, T \text{ and } i = 1, 2, 3),$$

where w_{1t} is the factor share of "blue collar" labor, w_{2t} is the factor share of "white collar" labor, and w_{3t} is the factor share of capital. The explanatory variables x_{1t}, x_{2t}, and x_{3t} are the three corresponding input quantities. Since $\sum_{i=1}^{3} w_{it} = 1$ for all t, this is a singular system, and only two of the three equations in (38) are estimated.

Specifications 9 to 11 are based on the annual Rotterdam system of demand equations, which may be written as:

$$\bar{w}_{it}\Delta \ln q_{it} = \alpha_i + \beta_i \sum_{j=1}^{n+1} \bar{w}_{jt}\Delta \ln q_{jt} + \sum_{j=1}^{n+1} \gamma_{ij}\Delta \ln p_{jt} + u_{it}$$
$$(\text{for } i = 1, \ldots, n+1 \text{ and } t = 1, \ldots, T),$$

where w_{it} is the budget share of commodity i; q_{it} is real expenditure on commodity i; p_{jt} is a retail price index for commodity j; and $\bar{w}_{it} = (w_{it} + w_{i,t-1})/2$. Again, this is a singular system, so that only n equations are estimated.

Finally, specification 12 is a quarterly and price-homogeneous version of the previous model:

$$\tilde{w}_{it}\Delta_4 \ln q_{it} = \alpha_i + \beta_i \sum_{j=1}^{n+1} \tilde{w}_{jt}\Delta_4 \ln q_{jt} + \sum_{j=1}^{n} \gamma_{ij}\Delta_4 \ln\left(\frac{p_{jt}}{p_{n+1,t}}\right) + u_{it}$$
$$(\text{for } i = 1, \ldots, n+1 \text{ and } t = 1, \ldots, T),$$

where Δ_4 is defined by $\Delta_4 y_t = y_t - y_{t-4}$, where $\tilde{w}_{it} = (w_{it} + w_{i,t-4})/2$, and with the following specification for the disturbance vector $u_t = (u_{1t}, \ldots, u_{nt})'$ under the alternative hypothesis that $R \neq O$:

$$u_t = R_1 u_{t-1} + R_4 u_{t-4} + v_t.$$

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