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Money-Consumption Substitution and Capital Accumulation on the Transition Path: Some Numerical Results

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ABSTRACT. – In a recent contribution VILLIEU [1992] extends the Sidrauski-Fischer model, considering various degrees of substitution between consumption and real money balances. He is able to characterise analytically the portions of the parameters space that imply an "anti-Tobin" effect. This paper shows, by means of numerical methods, that the procedure used by VILLIEU to prove his result is biased when the elasticity of substitution between consumption and real money balances is low.

Accumulation du capital et substituabilité entre consommation et encaisses réelles sur la trajectoire : quelques résultats numériques

RÉSUMÉ. – VILLIEU [1992] a examiné dans un modèle à la Sidrauski-Fischer la relation entre le taux de croissance de la masse monétaire et l'accumulation du capital et a analysé l'existence d'un effet « anti-Tobin ». Cette note montre que le résultat obtenu n'est pas correct lorsque l'élasticité de substitution entre consommation et encaisses réelles est inférieure à un.

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1 Introduction

In a recent contribution VILLIEU [1992] extends the Sidrauski-Fischer model, considering an instantaneous utility function that can encompass various degrees of substitution between consumption and real money balances.

In his model, variations in the rate of money growth cause changes in the time profile of the nominal interest rate. The consequences of such alterations are deeper than those highlighted by COHEN [1985, pp. 77-78 and 82-83], since they do not only shift expenditure over time, but they also affect its allocation between consumption and real money balances.

This expenditure allocation effect may result in a *negative* relation between the money growth rate and capital accumulation around the steady state. According to VILLIEU [1992, p. 83], such an “anti-Tobin” effect takes places either when the intertemporal elasticity of substitution is lower than one but higher than the elasticity of substitution between real money balances and consumption, or when the former elasticity exceeds unity but it is lower than the latter.

This paper argues that the procedure used by VILLIEU to prove his result is biased. We analyse the case with low values for the elasticity of substitution between consumption and real money balances mainly by means of numerical methods, and we show that the relation among the “Tobin effect”, the intertemporal elasticity of substitution and the elasticity of substitution between real money balances and consumption is reversed when the nominal interest rate is low.

In the second section we briefly outline the model and we set up the problem; in the third one we summarise the numerical results. The fourth section concludes.

2 The Basic Set-Up

VILLIEU [1992, p. 75 ff.] considers the intertemporal problem faced by a representative individual whose utility function is time separable:

$$\max_{\{c(t),\, m(t)\}} \int_0^{\infty} U(c(t),\, m(t))\, e^{-\beta t}\, dt \tag{1}$$

where $c(t)$ and $m(t)$ stand for, respectively, consumption and real money balances enjoyed by the representative agent at time t; β is the intertemporal preference rate.

The main innovation introduced by VILLIEU concerns the sub-utility function, which is assumed to be of the form [1, 2]:

$$U(c, m) = \frac{S}{S-1}\left[ac^{(\sigma-1)/\sigma} + (1-a)\, m^{(\sigma-1)/\sigma}\right]^{(\sigma(S-1))/(S(\sigma-1))}$$

where σ is the elasticity of substitution between consumption and real balances and S is the intertemporal substitution elasticity. It is assumed that $\sigma \in [0, \infty)$, $S \in (0, \infty)$ [3] and $a \in (0, 1)$. If σ is equal to one, this specification specialises to a Cobb-Douglas in consumption and money, *i.e.* to the case analysed by FISCHER [1979] and COHEN [1985].

Assuming constant population [4], the intertemporal budget constraint may be expressed as follows:

$$c + \overset{\circ}{k} + \frac{\overset{\circ}{M}}{p} = i\,k + w + \tau \tag{2}$$

where k represents per capita capital and M money in per capita nominal terms; p is the general price level, i is the real interest rate, w the labour income, and τ the lump sum transfer assigned to the representative individual by the Government [5]. The interest rate i, under perfect competition, is equal to the marginal productivity of capital, $f'(k)$, minus δ, the capital depreciation parameter.

The time derivative for the generic variable x is denoted by $\overset{\circ}{x}$.

The solution of the representative agent intertemporal problem (1) under the constraint (2) conveys various implications. Money is superneutral in the long run, *i.e.* the growth rate of money does not affect the steady state level of output. This SIDRAUSKI [1967]-type result is shared with a wide class of representative agent models and it is the root of the "friedmanite" money growth rule. In fact, given the complete ineffectiveness of the monetary policy on output, in the long run it is optimal to "satiate", if possible, the individuals with money and hence to reduce M at a rate equal to β (See FRIEDMAN, [1969]).

VILLIEU [1992, p. 75 ff.] analyses in detail problem (1)-(2), and he also shows that it is convenient to frame the dynamical system resulting from

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1. HARTMAN (1987, p. 475) assumes a utility function with a variable degree of substitution between consumption and real money balances but which is logarithmic over time. His results about the effects of the money growth rate on capital accumulation crucially depend on the introduction of distortionary income taxation.
2. From now on, we take as understood the time indexes whenever not confusing.
3. With $S = 0$, not only would the utility function degenerate, but also the determinant of the Jacobian of the dynamical system resulting from this model would be zero. (See below).
4. We implicitly introduce also the hypothesis of a zero growth rate for the representative family.
5. Notice that the representative agent should consider τ as independent from his real money holdings, otherwise money would be superneutral even on the transition path. (E.g. FISCHER [1979, p. 1434], DRAZEN [1981, p. 255], and ASAKO [1983, p. 1594]).

its solution in terms of consumption, capital and nominal interest rate. This choice allows us to linearise the dynamical system around the steady state obtaining the relatively simple jacobian (VILLIEU [1992, p. 85]):

(3)

$$J=\begin{bmatrix} 0 & \left[S\sigma\left(\frac{1+Ar^{*(\sigma-1)}}{S+\sigma Ar^{*(\sigma-1)}}\right)+\frac{S-\sigma}{S+\sigma Ar^{*(\sigma-1)}}\right]f''(k^*)\,c^* & \left(\frac{\sigma-S}{S+\sigma Ar^{*(\sigma-1)}}\right)c^* \\ -1 & \beta & 0 \\ 0 & \left(\frac{1+Ar^{*(\sigma-1)}}{S+\sigma Ar^{*(\sigma-1)}}\right)(S-1)\,f''(k^*)\,r^* & \left(\frac{1+Ar^{*(\sigma-1)}}{S+\sigma Ar^{*(\sigma-1)}}\right)r^* \end{bmatrix}$$

where r is the nominal interest rate, asterisks denote steady-state values of the corresponding variables, and $A \equiv (a/(1-a))^{\sigma}$.

From FISCHER [1979, p. 1434–5] (or VILLIEU [1992, p. 77-8]) we recall that, in the steady state, consumption is equal to net output and that the level of capital is such that its marginal productivity matches the intertemporal preference rate plus the capital depreciation parameter. Moreover, the nominal interest rate is equal to the intertemporal preference plus the rate of money growth. Hence, there is a one to one relation between r^* and ω, a fact that will be exploited later.

Two jumping variables are present in this model: consumption and nominal interest rate; only capital being a predetermined quantity. Hence, to establish saddlepath stability, we need to prove that there is only one negative eigenvalue. This can be done fairly easily, as $\mathrm{Det}\,(J) < 0$ and $\mathrm{Trace}\,(J) > 0$. Since the determinant is negative, we may have either one or three negative eigenvalues. Therefore, the sign of the trace enables us to establish that the negative eigenvalue is unique and that the explicit solution for the linearised differential equation for capital is:

$$k\,(t) = (k\,(0) - k^*)\,e^{\eta * t} + k^*$$

where η^* is the stable eigenvalue of J. Therefore, the effect of the money growth rate ω on the capital level is:

$$\frac{\partial k\,(t)}{\partial \omega} = \frac{\partial \eta^*}{\partial \omega}\,(k\,(t) - k^*)\,t \qquad (4)$$

When $k\,(t) < k^*$, $t \in (0,\, \infty)$ [6], we are in presence of the Tobin effect on the transition path if $\partial\eta^*/\partial\omega\,(\equiv \partial\eta^*/\partial r^*) < 0$. Therefore, it is necessary to check the sign of the effect of an increase in the money growth rate on the negative eigenvalue. The standard procedure, suggested by FISCHER [1979, p. 1437], exploits the total differential of the characteristic equation, evaluated at η^*:

$$\frac{\partial \mathbf{A}\,(\eta^*,\,\omega)}{\partial \omega}\,d\omega + \frac{\partial \mathbf{A}\,(\eta^*,\,\omega)}{\partial \eta}\,d\eta = 0$$

6. For t approaching infinity, we have $\lim_{t\to\infty} (k\,(t) - k^*)\,t = 0$.

where

$$(5)\quad \mathbf{A}(\eta^*,\,\omega) = (\eta^{*2} - \beta\eta^* + Sf''(k^*)\,c^*)\left[\left(\frac{(1+Ar^{*(\sigma-1)})\,r^*}{S+\sigma\,Ar^{*(\sigma-1)}}\right) - \eta^*\right] - \eta^*\,\frac{f''(k^*)\,c^*\,(S-1)\,(\sigma-S)}{S+\sigma\,Ar^{*(\sigma-1)}} = 0$$

Since η^* is the unique negative eigenvalue, $\partial\mathbf{A}(\eta^*,\,\omega)/\partial\eta < 0$, (see e.g. VILLIEU [1992, p. 86]) and our problem reduces to the evaluation of $\partial\mathbf{A}(\eta^*,\,\omega)/\partial\omega$. If it is negative, we are in presence of the Tobin effect, otherwise $\partial\eta^*/\partial\omega > 0$.

Differentiating (5) with respect to ω and substituting the resulting expression back into (5) itself we get:

$$\frac{\partial\mathbf{A}(\eta^*,\,\omega)}{\partial\omega} = \frac{f''(k^*)\,c^*\,\eta^*}{(1+Ar^{*(\sigma-1)})\,r^* - (S+\sigma\,Ar^{*(\sigma-1)})\,\eta^*} \times (S-1)\,(\sigma-S)\,X(S,\,\sigma,\,r^*)$$

where $X(S,\,\sigma,\,r^*) = 1 + \sigma\,Ar^{*(\sigma-1)} - (\sigma-1)\,\sigma\,Ar^{*(\sigma-2)}\,\eta^*(S,\,\sigma,\,r^*)$.

When $\sigma \geq 1\,X(\cdot)$ is positive; therefore $\mathrm{Sign}\,(\partial\mathbf{A}(\eta^*,\,\omega)/\partial\omega) = \mathrm{Sign}\,(S-1)\,(\sigma-S))$. The same conclusion immediately applies for $\sigma = 0$ [7]. The analysis of the interval $\sigma \in (0,\,1)$ is much harder, but it is relevant. A low but strictly positive elasticity of substitution between consumption and money fits, at the level of stylised facts, with most of the empirical contributions on the demand for money. In such studies, the estimated coefficients on interest rates are often low in absolute value but significant. For an example considering 27 countries, and among them the OECD ones but for Iceland, Luxembourg and Spain, see FAIR [1987].

VILLIEU [1992, pp. 87-89] investigates the sign of $X(\cdot)$ by inspecting the linearised dynamic behaviour of consumption and of the nominal interest rate. He concludes that $\mathrm{Sign}\,(\partial\mathbf{A}(\eta^*,\,\omega)/\partial\omega)$ is equal to $\mathrm{Sign}\,((S-1)\,(\sigma-S))$ also when $\sigma < 1$, and hence that $X(\cdot)$ is always positive. However, in his proof, he introduces the hypothesis that capital is given at its steady state level (VILLIEU, [1992, pp. 87-88]) [8]. This assumption induces a bias, since, for a given initial condition, the capital level is affected by η^* and hence by the variation of ω (equation (4)).

The analytical characterisation of the sign of $X(S,\,\sigma,\,r^*)$ is not an easy task, since, among the arguments of $X(S,\,\sigma,\,r^*)$, we have $\eta^*(S,\,\sigma,\,r^*)$. However, by inspecting (3) we notice that, if $S = \sigma$, $\eta^*(\sigma)$ becomes independent of r^* and equal to $(\beta - \sqrt{\beta^2 - 4f''(k^*)\,c^*\,\sigma})/2$. In this particular case, we show that $X(\sigma,\,r^*)$, for $\sigma \in (0,\,1)$, is negative when r^* approaches 0.

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7. This is the cash-in-advance case studied by ASAKO [1983].

8. This hypothesis is introduced computing $\partial c(t)/\partial\omega$, $\dfrac{\partial(\overset{\circ}{c}(t)/c(t))}{\partial\omega}$ and $\dfrac{\partial(\overset{\circ}{r}(t)/r(t))}{\partial\omega}$.

Reformulate $X(\sigma, r^*)$ as:

$$X(\sigma, r^*) = 1 + \sigma A \, r^{*(\sigma-1)} \left(1 - (\sigma - 1) \frac{\eta^*(\sigma)}{r^*}\right)$$

Since $\lim_{r^* \to 0^+} r^{*(\sigma-1)} = [+\infty]$ we have:

$$\begin{aligned}\lim_{r^* \to 0^+} X(\sigma, r^*) &= [+\infty] \left\{ \lim_{r^* \to 0^+} \left[\sigma A \left(1 - (\sigma - 1) \frac{\eta^*(\sigma)}{r^*}\right)\right]\right\} \\ &= [+\infty] \left\{ \sigma A \left[1 - (\sigma - 1) \lim_{r^* \to 0^+} \left(\frac{\eta^*(\sigma)}{r^*}\right)\right]\right\} \\ &= [-\infty].\end{aligned}$$

This result is limited but relevant, since, by continuity, we argue that $X(S, \sigma, r^*)$ is negative also for some $S \neq \sigma$ and for some $r^* > 0$. However, to asses the quantitative importance of the portion of the parameters space where $X(S, \sigma, r^*)$ is negative we need to resort to numerical techniques.

3 Numerical Results

Our computation strategy preliminary requires the adoption of a specific form for the production function. Normalising per capita labour supply to one, the simplest choice is:

(6) $$y = \phi k^\gamma - \delta k$$

where y is per capita output.

Therefore, the model contains eight parameters: S, σ, β, a, ω, ϕ, γ, δ. Notice, however, that the steady state values of consumption and of the second derivative of the production function always appear in equation (5) as a product. Since $\eta^*(\cdot)$ is the smallest root of that equation, the Cobb-Douglas specification (6) allows to substitute $(\beta+\delta)^2 (\gamma-1)/\gamma - \delta(\beta+\delta)\gamma/(\gamma-1)$ for $c^* f''(k^*)$ into (5), and $X(\cdot)$ is independent of ϕ.

Moreover, the value of $X(\cdot)$ does not prove to be very sensitive to a; therefore, choosing $a = 0.7$, and setting the rate of intertemporal preference, β, to 0.03 and the capital income share, γ, to 0.3, we can focus on S, σ and ω.

Assuming away capital depreciation and choosing a situation close to the friedmanite one, with $r^* = 0.001$, we plot $X(S, \sigma, 0.001)$ for $S \in \{0.5, 2\}$, and we obtain figure 1. When $S = 0.5$, $X(\cdot)$ is negative for $\sigma \in [0.000051, 0.931707]$; when $S = 2$, it is negative for $\sigma \in [0.000021, 0.971452]$.

In general, simulations show that the lower r^*, the larger the interval with negative $X(\cdot)$.

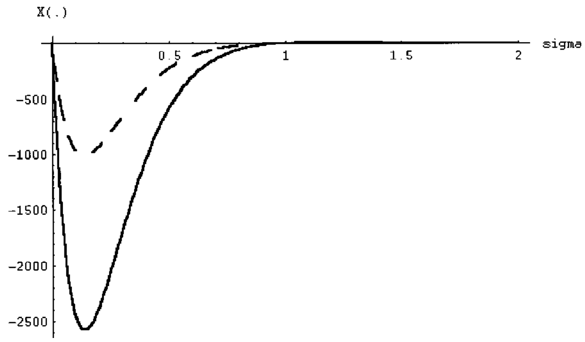

FIGURE 1

$X(S,\,\sigma,\,r^*)$ is plotted as a function of σ, for $r^* = 0.001$, when $S = 0.5$ (dashed line) and when $S = 2$ (continuous line).

It is remarkable that, if we accept the policy prescription provided by the model, VILLIEU's result is reversed in large portions of the parameters space, since $\text{Sign}\,(\partial \mathbf{A}\,(\eta^*,\,\omega)/\partial\omega)$ is opposite to $\text{Sign}\,((S-1)\,(\sigma - S))$ when $X(\cdot)$ is negative.

It is natural to investigate situations with more realistic nominal interest rates. We increased r^* by step of 0.001 starting from 0.001 and we first obtained a strictly positive surface, for $S \in (0,\,100)$ and $\sigma \in [0,\,1]$, with $\hat{r} = 0.066$. This proved to be the "critical value" for our parameter set, since a further increase in r^* always produced strictly positive surfaces, *i.e.* $\hat{r} = \min\{r^* :\, X(S,\,\sigma,\,r^*) > 0 | S \in (0,\,100),\,\sigma \in [0,\,1]\}$.

Table 1 shows the critical values, $\hat{r}$, for parameter a varying from 0.1 to 0.9 by steps of 0.1 and for $\delta \in \{0.0,\,0.1\}$.

TABLE 1

Critical Values of the Nominal Interest Rate.

Parameters values: $\beta = 0.03$, $\gamma = 0.3$			
$\delta = 0.0$		$\delta = 0.1$	
a	$\hat{r}$	a	$\hat{r}$
0.1	0.054	0.1	0.111
0.2	0.056	0.2	0.120
0.3	0.058	0.3	0.126
0.4	0.060	0.4	0.132
0.5	0.062	0.5	0.139
0.6	0.064	0.6	0.145
0.7	0.066	0.7	0.155
0.8	0.070	0.8	0.168
0.9	0.076	0.9	0.192

Note: S varies from 0.1 to 100 by steps of 1; σ varies from 0 to 1 by steps of 0.05.

Figure 2 provides some evidence about the speed of shrinking of the area where $X(\cdot)$ is negative, when $\hat{r}$ is increased. We draw the contour plots of $X(\cdot)$ at a height of zero for $r^* \in \{0.05, 0.10, 0.15\}$, for $a = 0.7$ and in the case of 10% capital depreciation.

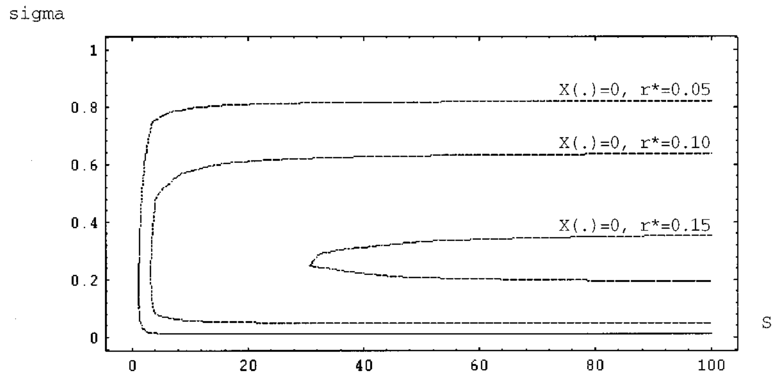

FIGURE 2

Contours of $X(S, \sigma, r^*)$ ***at an height equal to zero, for*** $r^* \in \{0.05, 0.10, 0.15\}$**.**

4 Final Remarks

This paper analyses a Sidrauski-type model where various degrees of substitutability between consumption and real money balances are taken into account.

We considered the case of a low elasticity of substitution between, real money balances and consumption, and we showed, by means of numerical calculations, that the relation among this parameter, the derivative of the stable eigenvalue with respect to the money growth rate and the intertemporal elasticity of substitution is not the simple one suggested by VILLIEU.

In particular, had the policy maker followed the friedmanite prescription delivered by the model, he would have obtained, around the steady state, results that are opposite to the one implied by VILLIEU's analysis in large portions of the parameters space.

• References

ASAKO, A. (1983). – "The Utility Function and The Superneutrality of Money on the Transition Path", *Econometrica*, 51, nr. 5, September, pp. 1593-96.

COHEN, D. (1985). – "Inflation, Wealth and Interest Rates in an Intertemporal Optimizing Model", *Journal of Monetary Economics*, 16, nr. 1, July, pp. 73-85.

DRAZEN, A. (1981). – "Inflation and Capital Accumulation Under a Finite Horizon", *Journal of Monetary Economics*, 8, nr. 2, September, pp. 247-60.

FAIR, R. C. (1987). – "International Evidence on the Demand for Money", *The Rewiew of Economics and Statistics*, 69, nr. 3, August, pp. 473-80.

FRIEDMAN, M. (1969). – "The Optimum Quantity of Money" in "*The Optimum Quantity of Money and Other Essays*", Aldine, Chicago, pp. 1-50.

FISCHER, S. (1979). – "Capital Accumulation on the Transition Path in a Monetary Optimizing Model", *Econometrica*, 47, nr. 6, November, pp. 1433-39.

HARTMAN, R. (1988). – "Money, Inflation and Investment", *Journal of Monetary Economics*, 22, nr. 3, November, pp. 473-84.

SIDRAUSKI, M. (1967). – "Rational Choice and Patterns of Growth in a Monetary Economics", *American Economic Review, Papers and Proceedings*, 57, nr. 2, May, pp. 534-44.

VILLIEU, P. (1992). – "Inflation et accumulation du capital: Le rôle de la substituibilité entre consommation et encaisses réelles", *Annales d'Économie et de Statistique*, nr. 27, July, pp. 73-89.