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Endogenous Growth and Poverty Traps in a Cournotian Model

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ABSTRACT. – We analyze the implications for the dynamics of capital accumulation of market power and endogenous demand elasticities, in an environment in which the latter are affected by the number of competitors in each industry. In equilibrium the interest rate increases as capital accumulates, even though the marginal product of capital is constant. Under standard assumptions two steady states and a balanced growth path exist, and the possibility of multiple equilibrium paths (for given initial conditions) arises. The latter feature is argued to match several empirical observations.

Croissance endogène et trappes de pauvreté dans un modèle à la Cournot

RÉSUMÉ. – On analyse les implications pour la dynamique d'accumulation du capital de variations endogènes du pouvoir de marché, dans un environnement où l'élasticité de la demande dépend du nombre d'entreprises présentes dans secteur. À l'équilibre, le taux d'intérêt croît au fur et à mesure que le capital s'accumule, bien que le produit marginal du capital soit constant. Sous des hypothèses standards, il existe deux états stationnaires ainsi qu'un chemin de croissance équilibrée ; à conditions initiales données, une multiplicité de chemins d'équilibre peuvent exister. Cette dernière caractéristique s'accorde avec plusieurs observations empiriques.

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1 Introduction

In an imperfectly competitive environment, investment decisions are affected by what firms correctly perceive as demand constraints: when considering whether to invest in an additional unit of capital any individual firm takes into account the price reduction that will be necessary in order to sell the marginal product. Those considerations are absent in the perfectly competitive environment assumed in the majority of growth models found in the literature, old and new [1].

In a series of recent papers we have tried to formalize some of those insights by studying the impact of departures from perfect competition in dynamic general equilibrium models of capital accumulation [2]. In this paper we pursue that idea by analyzing the growth dynamics of a nonconvex, imperfectly competitive version of the Ak model studied by REBELO [1991]. We introduce a nonconvexity by assuming the existence of an overhead requirement at the firm level. We depart from perfect competition by introducing a number of industries, each of which is characterized by a Cournotian market structure with free entry and exit. That framework generates two important relationships.

(i) the price-elasticity of demand that is relevant to the typical firm is a function of the extent of the competition it faces, as measured by the number of firms operating in its industry.

(ii) the number of firms in each industry is increasing in the aggregate capital stock.

From (i) and (ii) it follows that the price-elasticity of demand, as perceived by a typical firm, increases with the aggregate capital stock; in other words, the economy becomes more competitive as it grows. Since, ceteris paribus, the private return to investment is increasing in the price-elasticity of demand, the combination of (i) and (ii) yield a positive relationship between the aggregate capital stock and the private return to investment (and thus the market interest rate), given our assumption of a constant marginal product of capital. As we show below, that feature raises the possibility of multiple equilibria and development traps, both of which are absent from the perfectly competitive version of the Ak model.

The emergence of multiple equilibria in a growth model can also be found in other recent papers. Most existing models, however, are characterized by price taking behavior or constant markups, and rely on a different mechanism to generate multiple equilibria. Most available examples embed some kind of external increasing returns (see, e.g., BENHABIB and FARMER [1993], BOLDRIN [1993], ZILIBOTTI [1993]). Though our model contains

1. The importance of demand as a potential constraint on growth was emphasized by non classical economists like HOBSON, and Marxian economists like LUXEMBOURG or LENIN. In their analysis, the capitalists' inability to sell their output domestically at remunerative prices lies at the root of the imperialist policies pursued by european countries.
2. See, e.g., our previous work in GALÍ [1993*a, b*, 1994] and ZILIBOTTI [1994].

a nonconvexity the latter is just a device to bound the number of firms operating in a given industry at any given point in time, but does not play a role in generating the possibility of multiple equilibria. That possibility is instead a consequence of the market structure and the effect of entry on the degree of competition.

The plan of the paper is as follows. In section 2 we set up the basic model. Section 3 derives the dynamical system characterizing an equilibrium. Section 4 discusses the existence and multiplicity of stationary equilibria. Section 4 analyzes the equilibrium dynamics. Section 6 concludes. A technical appendix at the end gathers the key technical propositions and proofs.

2 The Model

2.1. Final Goods Sector

There is a single final good produced by a representative firm, which has access to a constant returns technology represented by the production function

$$Y = \left\{ \int_0^1 X_j^{(\varepsilon-1)/\varepsilon} dj \right\}^{\varepsilon/(\varepsilon-1)} \tag{1}$$

where Y is the quantity of final output, and X_j, $j \in (0, 1)$, are the quantities of the different intermediate inputs used, with the range of input types normalized to have measure one.

The firm purchases the necessary inputs from firms in the intermediate goods sector at unit prices p_j, $j \in (0, 1)$, and sells its output to households at a price P_y. It is assumed to behave competitively both as a buyer and as a seller. At any given point in time, it seeks to maximize its profit,

$$\pi^F = P_y Y - \int_0^1 p_j X_j dj$$

subject to (1). Letting $E \equiv \int_0^1 p_j X_j dj$, the solution to that problem yields the input demand schedules

$$X_j = (p_j / P_x)^{-\varepsilon} (E/P_x) \qquad j \in (0, 1) \tag{2}$$

where $P_x \equiv \left\{ \int_0^1 p_j^{1-\varepsilon} dj \right\}^{1/(1-\varepsilon)}$. Substitution of (2) into (1) yields

$$Y = E/P_x \tag{3}$$

which, in turn, allows us to write $\pi^F = (P_y - P_x) Y$. Competitive behavior implies that both P_x and P_y are taken as given by the firm. Thus,

$$P_x = P_y \equiv P \tag{4}$$

must hold in equilibrium (otherwise the supply of final goods would be zero or infinite), implying that zero profits are made in this sector.

2.2. **Intermediate Goods Sector**

There is a continuum of industries represented by the unit interval. Each industry is made up of a finite number of firms which produce a *homogeneous* good that is used by the final sector as an input. An intermediate firm in industry j has access to a technology, described by a production function

$$x_j = A\,(k_j - \phi), \qquad k_j \geq \phi \tag{5}$$

where x is output, k is capital, and ϕ is overhead capital. Thus, and as in REBELO [1991], we assume that the marginal product of capital (A) is a positive constant [3].

At each point in time, a typical firm in industry j maximizes its profits

$$\pi_j^I = p_j x_j - q\,k_j$$

subject to (5) and the demand constraint given by

$$p_j \leq P\,\{E/P\;X_j\}^{1/\varepsilon} = P\,\{E/P\,[x_j + (n_j - 1)\,\overline{x}_j]\}^{1/\varepsilon}$$

while taking as given the rental cost of capital q, the number of firms in the industry (n_j) and the average output produced by the remaining firms in the industry $(\overline{x}_j)$, as well as other relevant economy-wide variables (namely, E and P). Thus, firms are assumed to behave in a Cournot fashion within each industry, each recognizing its impact on industry output and prices, but perceiving itself to be negligible in the economy as a whole.

The first order conditions for that problem are

$$p_j\,(1 - 1/\xi_j) = q/A \tag{6}$$

where ξ_j denotes the price-elasticity of the demand schedule faced by an individual firm in industry j, which is given by

$$\xi_j = \varepsilon\,[1 + (n_j - 1)\,(\overline{x}_j/x_j)]$$

We assume $\xi_j > 1$ at all times; below we discuss the conditions that guarantee that such a constraint holds. Notice that q/A measures firm j's marginal cost, so $(1 - 1/\xi_j)$ can be interpreted as the reciprocal of the optimal markup.

In a symmetric equilibrium all firms in all industries produce the same quantity (x) and charge the same price (p). Furthermore, the number of firms will be the same in all industries. We denote that number by N. It follows that $P = p$, $X = N\,x$, and

$$\xi = \varepsilon\,N$$

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3. Of course, in our case this is true as long as $k_j \geq \phi$; otherwise the marginal product of capital is zero.

must hold in such an equilibrium. The latter result implies a proportionality between the price-elasticity of demand and the number of active firms. Given (6), it follows that

$$P\,(1 - 1/\varepsilon N) = q/A \tag{7}$$

must hold in equilibrium. In addition, output per firm will be given by $x = A\,(K/N - \phi)$. Assuming $K - \phi N > 0$ (a necessary condition to generate positive output), we can express the profit obtained by the typical firm in the intermediate sector as

$$\pi^{I} = (\varepsilon N/(\varepsilon N - 1))\, q\,(K/N - \phi) - q\, K/N$$

Notice that the entry of new firms, which corresponds to an increase in N, always reduces the level of individual profits, given the aggregate capital stock. Under the assumptions of free entry and zero profits we can pin down the number of firms operating in each industry as a function of the aggregate capital stock (K). Ignoring integer constraints, we have

$$N = \sqrt{K/\varepsilon\phi} \tag{8}$$

Thus, we see that the number of firms is increasing in the aggregate capital stock. Furthermore, given (8), we can show that the restriction $\xi = \varepsilon\, N > 1$ is equivalent (in equilibrium) to the nonnegative output condition, *i.e.* $K - \phi\, N \geq 0$ [4]. Furthermore, both conditions can be expressed in the form of a lower bound for the aggregate capital stock:

$$K > \phi/\varepsilon \tag{9}$$

Whenever $K \leq \phi/\varepsilon$, an equilibrium with positive output does not exist. Below we discuss what happens in that case.

2.3. Consumers

We assume an infinite-lived representative consumer who seeks to maximize

$$\int_0^{\infty} U\,(C\,(t))\, \exp\,(-\rho t)\, dt \tag{10}$$

where ρ is the time discount rate, U is the instantaneous utility function, and C is the quantity of the final good consumed. We specify U to be of the form

$$U\,(C) = (\sigma/(\sigma - 1))\, C^{(\sigma-1)/\sigma}, \qquad \text{for} \quad \sigma \neq 1$$
$$U\,(C) = \log\, C, \qquad \text{for} \quad \sigma \neq 1$$

where $\sigma > 0$ is the intertemporal elasticity of substitution.

4. To see this, notice that whenever $K \geq \phi N$, (8) implies $N \geq \sqrt{N/\varepsilon}$ and, consequently, $\xi = \varepsilon\, N \geq 1$.

Each period the household rents its capital holdings to firms in the intermediate sector at a rental price q. That income is used to purchase the final good, at a price P. Those purchases are added to a constant flow ω of the final good that the consumer gets as an endowment. We think of ω as the output of some basic household technology that does not use capital. The resulting quantity is then split between investment and consumption [5]. Hence, the consumer's dynamic budget constraint takes the form

$$\dot{K} = r\,K + \omega - C \tag{11}$$

where $r \equiv q/P$ denotes the instantaneous return to capital or, equivalently, the "shadow" interest rate on a riskless asset. Unless necessary, the obvious time dependence of all the variables is not made explicit hereafter, in order to ease our notation.

Our consumer is assumed to behave competitively, taking the rental price and the price of the final good (and, consequently, the interest rate) as given. Thus his problem consists in choosing a path for C and K that maximizes (10) subject to (11), the nonnegativity constraints $K \geq 0$ and $C \geq 0$, and an initial condition for the capital stock, given the path of q and P. The Euler equation corresponding to that optimal control problem is

$$\dot{C}/C \geq \sigma\,(r - \rho) \tag{12}$$

which holds as an strict inequality whenever the nonnegativity constraint on capital is binding, and $K = 0$.

The associated tranversality condition is given by

$$\lim_{T \to \infty} K\,(T)\,C\,(T)^{-1/\sigma}\,\exp\left(-\rho T\right) = 0. \tag{13}$$

3 Equilibrium

In equilibrium the rental income accruing to the representative consumer equals the revenue generated by the intermediate sector, which in turn equals the output of the final good Y. Formally,

$$q\,K/P = Nx = A\,(K - \phi N) = A\,K\,(1 - \sqrt{\phi/K\,\varepsilon}) \tag{14}$$

Using (14) we can write the interest rate $r \equiv q/P$ as a function of the aggregate capital stock:

$$r\,(K) = A\,(1 - \sqrt{\phi/K\,\varepsilon}) \tag{15}$$

5. In order to simplify our notation we assume a zero depreciation rate.

Substituting (15) into (11) and (12), we obtain the system of differential equations

$$\dot{K} = A\,K\,(1 - \sqrt{\phi/K\,\varepsilon}) + \omega - C \tag{16}$$

$$\dot{C} \geq \sigma\,C\,[A\,(1 - \sqrt{\phi/K\,\varepsilon}) - \rho] \tag{17}$$

We can now define an equilibrium path of our model economy as a trajectory of the dynamical system (16)-(17), that satisfies the initial condition $K\,(0) = K_0$, the transversality condition (13), and the nonnegativity constraint $C\,(t) \geq 0$ for all $t \geq 0$ such that $K\,(t) > \phi/\varepsilon$. Whenever $K\,(t) \leq \phi/\varepsilon$, the equilibrium dynamics are given by

$$\dot{K} = \omega - C \tag{18}$$

$$\dot{C} \geq -\sigma\,C\,\rho \tag{19}$$

which can be viewed as a particular case of (11) and (12), with $r = 0$. Thus, whenever $K\,(t) \leq \phi/\varepsilon$, no firms are operating and consumers are implicitly assumed to "invest" their capital in a storage technology (with no depreciation) yielding a zero net return to investment.

Next we turn to a characterization of equilibrium paths for K and C. We start by analyzing the existence and properties of stationary equilibria. Given an equilibrium path for K and C it is straightforward to obtain the corresponding paths for aggregate output, the interest rate and the number of operating firms, using (8), (14), and (15).

4 Stationary Equilibria and Balanced Growth Path

Our economy has two stationary equilibria. One of them is given by the corner allocation $(K,\, C) = (0,\, \omega)$. We refer to it as the autarky steady state. In that equilibrium there are no active firms in any industry, consumers have eaten up all their capital holdings, and they just consume their current endowment.

In addition, and under the assumption

$$A > \rho \tag{20}$$

there exists an interior stationary equilibrium in which the interest rate r is equal to the discount rate ρ, and which is given by

$$K^* = (\phi/\varepsilon)\,[A/(A-\rho)]^2 > (\phi/\varepsilon) \tag{21}$$

$$C^* = \omega + \rho\,(\phi/\varepsilon)\,[A/(A-\rho)]^2 \tag{22}$$

As $K \to +\infty$, the interest rate converges to A and aggregate output becomes linear in the aggregate capital stock, *i.e.*, $Y/K \to A$. Furthermore, the endowment flow ω becomes negligible relative to the size of the economy. Thus, asymptotically, there exists an equilibrium path characterized by sustained growth and which corresponds to the equilibrium path of the standard AK model (see REBELO [1991]. Along that path, C, K, and Y grow at a constant rate γ, given by

$$\gamma = \sigma (A - \rho) \tag{23}$$

with the level of consumption being given by

$$C = K [(1 - \sigma) A + \sigma\rho] \tag{24}$$

A necessary and sufficient condition for that balanced growth path to be an equilibrium is given by

$$\sigma < A/(A - \rho) \tag{25}$$

which guarantees that the consumer's objective function is well defined, the transversality condition holds, and C is positive [6].

5 Equilibrium Dynamics

Figure 1 shows the phase diagram corresponding to the dynamical system (16)-(17) and (18)-(19). The $\dot{K} = 0$ schedule corresponds to the horizontal locus $C = \omega$ for $K \leq \phi/\varepsilon$, is increasing and convex for $K > \phi/\varepsilon$. Trajectories below (above) that schedule correspond to an increasing (decreasing) aggregate capital stock, as represented by the horizontal arrows. The $\dot{C} = 0$ schedule is given, in addition to the K-axis, by a vertical line at K^*, the unique solution to $r(K) = \rho$ under our assumptions. Trajectories to the left (right) of that schedule are associated with decreasing (increasing) consumption levels.

We also draw a ray from the origin that corresponds to the C/K ratio associated with the asymptotic balanced growth path, given by (24). We denote that ray by $(C/K)_\infty$.

As shown in the technical appendix, there exists a unique trajectory that converges to the autarky steady state $(0, \omega)$ while satisfying all the equilibrium conditions. That trajectory approaches $(0, \omega)$ from the right and

6. Notice that along the asymptotic balanced growth path $\lim_{T\to\infty} (\dot{K}/K) - (1/\sigma) (\dot{C}/C) - \rho = \sigma(A-\rho)-A < 0$, where the inequality follows from (25). That guarantees that the transversality condition is satisfied along that asymptotic path.

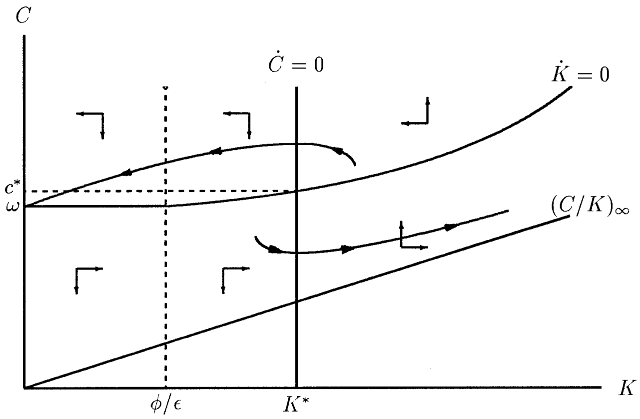

FIGURE 1

Phase Diagram.

above, after having crossed the $\dot{K} = 0$ locus and the $K = \phi/\varepsilon$ locus (in that order). That trajectory could in principle have originated in (K^*, C^*) or at some point on the C axis below $(0, \omega)$.

In addition to the path converging to $(0, \omega)$, there exists an additional trajectory satisfying all the equilibrium conditions and which converges, asymptotically, to the balanced growth path associated with the perfectly competitive AK model. In terms of our diagram, that trajectory also has its origin in (K^*, C^*) or at some point on the C axis below $(0, \omega)$, and it asymptotes to the $(C/K)_\infty$ ray as $K \to +\infty$. In the technical appendix we prove the existence and uniqueness of such a trajectory.

Clearly, we need to know the « full shape » of the two trajectories just described if we want to characterize the number and behavior of trajectories that, in addition to satisfying (16)-(20), (13) and the nonnegativity constraints, are consistent with *a given initial condition*. Unfortunately, and given the nonlinear nature of the dynamical system involved, a full characterization of the latter for all possible parameter values is nontrivial and, in any event, beyond the scope of this paper. Our strategy consists, instead, in calibrating the model using what we view as plausible parameter values and then tracing the associated equilibrium trajectories using numerical methods.

We calibrate the model as follows. We set $A = 0.06$, roughly the average annual real return on equity in the postwar U.S. We choose a value for the discount rate ρ equal to 0.04. In our benchmark case, we set $\sigma = 1$, which corresponds to logarithmic preferences, and which, combined with the previous settings, implies an asymptotic growth rate along the balanced growth path of 2 percent ($\gamma = 0.02$), roughly the average rate of growth of per capita income in the postwar U.S. economy. We set $\omega = 100$, a choice with no major significance beyond that of a normalization. We choose a value of 80 for parameter ϕ, *i.e.*, we assume that the overhead that would be required to set up one firm in every industry is equivalent to 80 percent of the aggregate consumption level in the autarky state [7]. We set ($\varepsilon = 2$; such a value implies a markup of 2 (*i.e.*, price equals twice the marginal cost) when there is a single firm in each industry.

Figure 2 portrays the equilibrium paths of our calibrated economy, drawn as solid lines with arrows. We can distinguish three relevant regions for the initial capital stock $K(0)$. If $K(0) \in (0, a)$ the equilibrium is unique and converges to the autarky steady state. Along that equilibrium path consumers eat up their endowment ω and part of their capital holdings, and end up consuming only ω. Any economy whose initial capital stock is in this region is effectively in a "poverty trap": the low capital stock is not sufficient to support the minimum number of firms that would be required to make the demand elasticity (and, thus, the return to investment) high enough to support further capital accumulation and entry. If $K(0) \in (b, +\infty)$ the equilibrium is also unique, but in this case it involves unbounded capital accumulation and convergence to an asymptotic path in which consumption, capital, and output all grow at a 2 percent annual rate. Intuitively, the high initial capital stock guarantees a high degree of competition, a high demand elasticity and, consequently, a return to investment that is sufficient to support further capital accumulation, which will in turn increase the return to investment even further, reinforcing the growth process. As is evident in Figure 2, multiple equilibrium paths coexist whenever $K(0) \in [a, b]$. Each path is characterized by perfect foresight, *i.e.* is fully consistent with agents' initial expectations. Furthermore, some of those paths converge to the asymptotic balanced growth path, while some lead the economy into the poverty trap region. Thus, if agents anticipate that the economy as a whole will follow the trajectory converging to the asymptotic balanced growth path, the expectation of higher future returns to investment makes

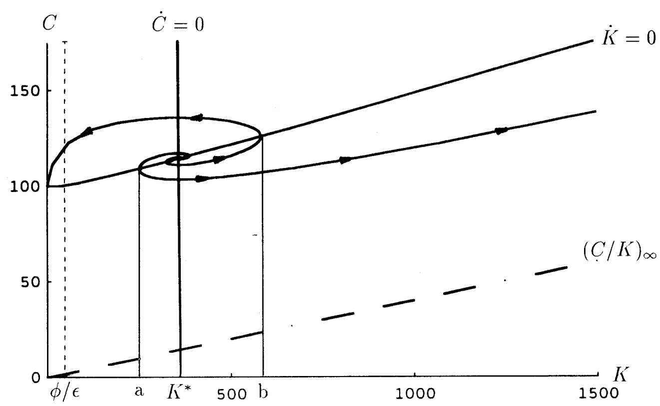

FIGURE 2

$\sigma = 1$.

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7. Recall that the set of industries has measure one in our model. Admittedly, this choice is rather arbitrary, but none of the qualitative results discussed below were significantly affected by it.

it optimal for any *individual* agent to make investment decisions that are consistent, in the aggregate, with the foreseen trajectory. Alternatively, if agents are pessimistic and come to believe that the economy will eventually experience negative savings which will take it to the "poverty trap" region, their anticipation of low future investment returns resulting from the decline in competition will affect their saving decisions in a self-fulfilling fashion. Whether the economy converges to the path with unbounded growth or the to the "autarky" state depends on which of those equilibrium paths is actually chosen. In other words, the initial conditions are not enough in that case to pin down the outcome that will be observed, and the latter can differ for economies that have identical initial conditions.

Figures 3 and 4 illustrate some of the qualitative changes in the equilibrium dynamics that are brought about by a sufficiently large increase or decrease in the intertemporal elasticity of substitution σ, while keeping the remaining parameters at their benchmark settings.

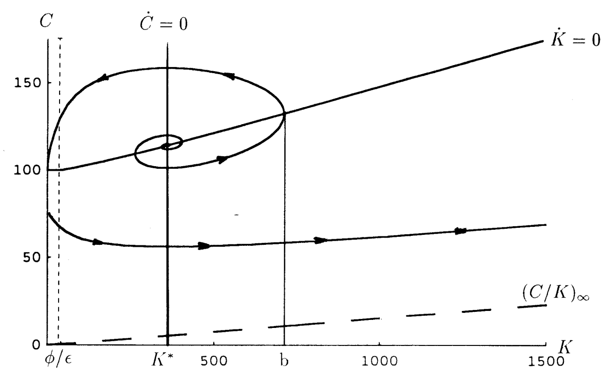

FIGURE 3

$\sigma = 2.2.$

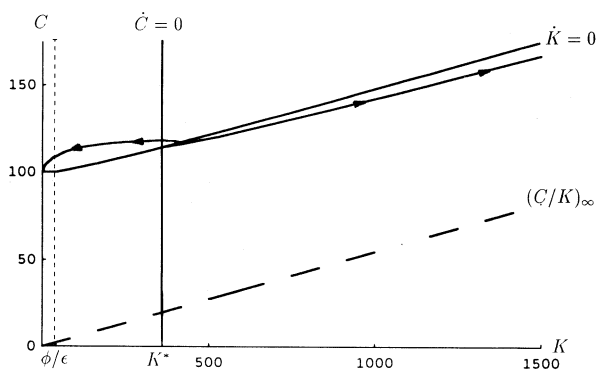

FIGURE 4

$\sigma = 0.25.$

In Figure 3, we have set $\sigma = 2.2$. As in the previous figure, there exists a threshold level b, such that if $K(0) > b$ the equilibrium is unique and converges asymptotically to the balanced growth path. In contrast, now there is no minimum capital stock necessary for an equilibrium path with unbounded growth to exist; in other words, no matter how low an economy's initial capital stock is, it can always "hope" to take off and enjoy unlimited growth subsequently. The economy may be poor, but not because it is in a poverty trap.

Figure 4 shows the equilibrium paths consistent with $\sigma = 0.25$. As a comparison of Figures 2 and 4 makes clear, a reduction in consumers' willingness to substitute intertemporally shrinks the size of the interval of $K(0)$ values for which there exists a multiplicity of equilibria. In the case portrayed, that interval is already hard to see, and would eventually vanish for a low enough σ.

The possibility of multiple equilibrium paths in our model results from the "complementarity" among individual investment decisions caused by an equilibrium interest rate schedule which is increasing in the aggregate capital stock. In that case the expectation that the economy as whole will follow a path of high investment, with associated high entry rates and low markups, raises the anticipated *private* return on savings, thus inducing individual savings decisions consistent with the initial expectations. Needless to say, many other authors have developed models that exhibit, in their reduced form, similar macroeconomic complementarities, and where the latter are also a source of multiple equilibria. From this viewpoint, our model is formally related to some recent work that aims at examining the equilibrium implications of external increasing returns in the context of dynamic models with capital accumulation and which often find such technologies to be a potential source of multiple equilibria (see, e.g., BENHABIB and FARMER [1993], BOLDRIN [1993], ZILIBOTTI [1993]) [8]. In contrast, our work points to a completely different (and, in our view, more plausible) source of multiplicity: the interaction between entry and exit, the elasticity of demand, and the return to investment, in an imperfectly competitive environment.

6 Summary and Conclusion

In this paper we have developed a dynamic general equilibrium model of capital accumulation with increasing returns and imperfect competition. The model is characterized by a continuum of industries, and Cournot competition with free entry and exit within each industry. Given a constant marginal product of capital at the firm level, such a framework generates

8. See also KRUGMAN [1991] and MATSUYAMA [1991] for a different class of dynamic models with multiple equilibria.

an equilibrium interest rate schedule which is increasing in the aggregate capital stock. The latter feature implies the existence of two steady states, one of which is always unstable, while the other is characterized by an autarky allocation with zero capital. Most interestingly, the model has been shown to yield, for plausible parameter values and for some range of initial conditions, multiple equilibrium paths, some of which drive the economy to the autarky state, while some other converge to the asymptotic balanced growth path. For some parameter values the possibility of a "poverty trap" emerges: for levels of the initial aggregate capital stock below a certain threshold, the equilibrium is unique and converges to the autarky, zero capital state.

The equilibrium dynamics of our model are consistent with a variety of empirical observations that are at odds with the predictions of the neoclassical model (CASS [1965], KOOPMANS [1965]). First, and like any other model with "endogenous growth", our equilibrium dynamics are consistent with the absence of unconditional convergence in per capita income at the worldwide level (e.g., BARRO [1991]). Second, the multiplicity of equilibrium paths for some range of initial conditions can potentially explain the sudden take-off episodes that lead to a growing income gap between economies that were similar at some point, as illustrated by the examples of South Korea and the Philippines (LUCAS [1993]). Third, and in a setting with international capital mobility our model could potentially explain the failure of capital to move from rich to poor countries (e.g., LUCAS [1990]) [9].

9. In fact, our model would predict capital flight from poor, low interest rate economies to rich, high interest rate economies.

A. **Analysis of Local Stability Properties**

Linearization of (16) and (17) around the stationary equilibrium (K^*, C^*) yields the dynamical system:

$$\begin{bmatrix} \dot{K} \\ \dot{C} \end{bmatrix} = \begin{bmatrix} (1/2)\,(A+\rho) & -1 \\ (\sigma/2)\,(A-\rho)\,(C_2^*/K_2^*) & 0 \end{bmatrix} \begin{bmatrix} K-K^* \\ C-C^* \end{bmatrix}$$

where $(C^*/K^*) = \rho + (\omega\varepsilon/\phi)[1-(\rho/(A)]^2$. The corresponding eigenvalues are

$$\lambda = (1/4)\,\{(A+\rho) \pm \sqrt{(A+\rho)^2 - 8\sigma\,(A-\rho)\,(C^*/K^*)}\}$$

We can distinguish the following two generic cases regarding the stability properties of the system around (K^*, C^*):

Case 1: $\sigma < (1/8)\,[(A+\rho)^2/(A-\rho)\,(C^*/K^*)]$. In this case both eigenvalues are real and positive, so (K^*, C^*) is an *unstable node*.

Case 2: $\sigma > (1/8)\,[(A+\rho)^2/(A-\rho)\,(C^*/K^*)]$. In this case the eigenvalues form a complex conjugate pair, with positive real part, so (K^*, C^*) is an *unstable focus*.

For $K \leq \phi/\varepsilon$, the dynamics are given by the linear system (18)-(19). The eigenvalues corresponding to that system are $\lambda_1 = -\sigma\rho$, and $\lambda_2 = 0$. Let t^* be such that $K\,(t^*) = \phi/\varepsilon$, with $C\,(t^*) > \omega$ (*i.e.*, at $t = t^*$ the economy enters the $K \leq \phi/\varepsilon$ region). Solving (18)-(19) for $t \geq t^*$ yields:

$$K(t) = [(\phi/\varepsilon) - (1/\sigma\rho)C(t^*)] + \omega(t-t^*) + (1/\sigma\rho)C(t^*)\exp[-\sigma\rho(t-t^*)]$$
$$C\,(t) = C\,(t^*)\,\exp\,[-\sigma\rho\,(t-t^*)]$$

It is not difficult to show the existence of a unique $C\,(t^*)$ value such that $K\,(T) = 0$ and $C\,(T) = \omega$ for some T. More precisely, $C\,(t^*) = \omega + (\sigma\rho\phi/\varepsilon) + \sigma\rho\omega\,(T-t^*)$ where $T > 0$ is implicitly determined by

$$\exp\,[\sigma\rho\,(T-t^*)] = 1 + (\sigma\rho\phi/\omega\varepsilon) + \sigma\rho\,(T-t^*)$$

That $C\,(t^*)$ level determines a unique trajectory that remains in the $K \leq \phi/\varepsilon$ region and converges to $(K, C) = (0, \omega)$ in finite time. Any trajectory that crosses the $K = \phi/\varepsilon$ axis above that unique $C\,(t^*)$ level will eventually hit the $K = 0$ axis above ω, implying a "jump" in consumption that violates the consumer's Euler equation. Trajectories that cross the $K = \phi/\varepsilon$ axis below $C\,(t^*)$ cross the $\dot{K} = 0$ locus in finite time, and later leave the $K \leq \phi/\varepsilon$.

B. **Equilibrium Trajectories with Self-Sustained Growth Converge to the Asymptotic Balanced Growth Path**

The next two propositions show that trajectories with unbounded growth are consistent with equilibrium only if they converge to the balanced growth path.

PROPOSITION 1: Trajectories with sustained growth are characterized by either lim $\lim_{T\to+\infty} C(T)/K(T) = (1-\sigma)A + \sigma\rho$ (balanced growth case, see (23)), or $\lim_{T\to+\infty} C(T)/K(T) = 0$.

Proof: Define $\zeta \equiv \log(C/K)$. Then (16) can be rewritten as:

$$(A1) \qquad \dot{\zeta} = (\dot{C}/C) - (\dot{K} - K) = \Gamma(K) - \sigma\rho + \exp(\zeta)$$

where $\Gamma(K) = (\sigma-1)r(K) - (\omega/K)$. Also, let $\zeta^* = \log[A(1-\sigma) + \sigma\rho]$. Consider trajectories which remain, starting from some t, in the region of the phase portrait where K grows unboundedly (see Figure 1). Using $\lim_{K\to+\infty} \Gamma(K) = (\sigma-1)A$, it follows that there are three types of solutions to (T1) within the class considered:

(i) $\lim_{T\to+\infty} \dot{\zeta} = 0$, $\lim_{T\to+\infty} \zeta = \zeta^*$

(ii) $\lim_{T\to+\infty} \dot{\zeta} = (\sigma-1)A - \sigma\rho < 0$, $\lim_{T\to+\infty} \zeta = -\infty$

(iii) $\lim_{T\to+\infty} \dot{\zeta} = +\infty$, $\lim_{T\to+\infty} \zeta = +\infty$

where the inequality in (ii) follows from (25). Notice, however, that (iii) is not feasible for it implies that $\lim_{T\to+\infty} (C(T)/K(T)) = \lim_{T\to+\infty} A(C(T)/Y(T) = +\infty$, which is not possible along trajectories with sustained growth. The proposition then follows from the definition of ζ. $\square$

Notice that solution (ii) above implies that the trajectory approaches that K-axis of Figure 1, as $T \to +\infty$. Next we prove that only trajectories of type (i) satisfy the transversality condition.

PROPOSITION 2: Consider trajectories with sustained growth $(K \to +\infty)$ which satisfy equations (16), (17), and the nonnegativity constraints. Such trajectories satisfy the transversality condition if and only if $\lim_{T\to+\infty} C(T)/K(T) = A(1-\sigma) + \sigma\rho$, *i.e.* if and only if they converge to the unique balanced growth path.

Proof: We have already shown that, under our assumptions, any trajectory that converges to the balanced growth path satisfies the transversality condition and is an equilibrium. Given Proposition 1 we just need to show that any trajectory such that $\lim_{T\to+\infty} C(T)/K(T) = 0$ violates the transversality condition. Notice that under such a hypothesis $\lim_{T\to+\infty} (\dot{K}/K) = A$, so that

$$\begin{aligned} &\lim_{T\to+\infty} K(T)\,C(T)^{-1/\sigma} \exp(-\rho T) \\ &\quad = \phi \exp[(-\rho - (1/\sigma)\sigma(A-\rho) + A)T] = \phi, \end{aligned}$$

where

$$\phi \equiv K\,(0)\,C\,(0)^{-1/\sigma}\lim_{T\to+\infty}\exp\left\{\int_0^T((\omega - C\,(t))/K\,(t))\,dt\right\} > 0.$$

Thus, the transversality condition is violated when $\lim_{T\to+\infty} C(T)/K(T) = 0$.

□

C. **Existence and Uniqueness of the Trajectory Converging to the Asymptotic Balanced Growth Path**

Next we show that at least one equilibrium exists for any initial condition, and that the equilibrium trajectory with self-sustained growth is unique. The proofs exploit an argument used in ZILIBOTTI [1993], to which we refer the interested reader for further discussion.

PROPOSITION 3: Under assumptions (20) and (25), for any initial condition K (0) there exists at least one consumption level $C\,(0) > 0$ such that $(K\,(0),\,C\,(0))$ belongs to an equilibrium trajectory.

Proof: The existence of an equilibrium for all $K(0) \leq K^*$ has been proved in the first part of this appendix. Here we prove that for all $K\,(0) > K^*$ there exists an equilibrium trajectory with sustained growth. (20) and (25) guarantee that an asymptotic balanced growth path exists and that it is consistent with positive growth and consumption.

It is easy to check that existence is guaranteed for *some* $K\,(0)$. Consider a trajectory that solves the differential equation (TA1) with boundary condition $\lim_{T\to+\infty}\zeta\,(T) = \zeta^*$ and given an initial condition $K\,(0)$. By continuity, and given that $\lim_{K\to+\infty}\Gamma\,(K) = (\sigma - 1)\,A$, there is always a large enough value of $K\,(0)$ such that an equilibrium trajectory such that (C/K) remains arbitrarily close to $(C/K)_\infty$ exists.

Case 1: $\sigma \geq 1$. In this case $\Gamma' > 0$. First we show that $\dot{\zeta} < 0$ in this case. Suppose that $\dot{\zeta} \geq 0$ at some t. Then we would have,

$$(A2) \qquad d\dot{\zeta}/dt = \Gamma'\,(K)\,\dot{K} + \exp\,(\zeta)\,\dot{\zeta} > 0$$

and, hence, $\lim_{T\to+\infty}\zeta = +\infty$, thus violating the boundary condition. Thus, if $\sigma \geq 1$, ζ must be nonincreasing for any path converging to ζ^*. Since $\zeta^* > 0$, it follows that $\zeta > 0$ and, hence, $C > 0$.

Case 2: $\sigma < 1$. Consider the condition $\dot{\zeta} = 0$. That condition holds whenever $\zeta = \log\,(\sigma\rho - \Gamma\,(K)) \equiv M\,(K) > 0$. We claim that along any trajectory satisfying the differential equation (TA1) and the boundary condition $\lim_{T\to+\infty}\zeta\,(T) = \zeta^*$ we must have $\zeta\,(t) \geq \inf_K M\,(K)$. Suppose that this was violated for some $t = v$. Then $\dot{\zeta} < 0$ for all $t > v$ (since $\zeta\,(t) < M\,(K\,(t)) \Leftrightarrow \dot{\zeta} < 0$) and the boundary condition would not be met. This is sufficient to complete the proof of the proposition since it implies that both ζ and, hence, C are positive at all $t > 0$, for any $K(0)$, when $\sigma < 1$. $\square$

PROPOSITION 4: Under assumptions (20) and (25), the equilibrium trajectory with sustained growth is unique.

Proof: Define $S(K,C) \equiv \dot{C}/\dot{K} = [C(t)\sigma(r(K)-\rho)]/[r(K)K+\omega-C]$. Geometrically, $S(K,C)$ is the slope of the trajectory of the dynamical system at (K, C). Notice that whenever $\dot{K} > 0$ and $\dot{C} > 0$ we have $\partial S/\partial C > 0$, implying that any pair of trajectories characterized by unbounded growth cannot converge to each other. It follows that there cannot exist two different trajectories that converge to the asymptotic balanced growth path. $\square$

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