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Mixed Duopoly Under Vertical Differentiation

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ABSTRACT. – In this paper we study a vertically differentiated duopoly market with a profit maximizing firm (private firm) and a total surplus maximizing firm (public firm). The technological conditions are assumed identical for both firms and are described by unit costs which are constant with respect to quantity, though increasing in quality. No specific form is given to the relation between unit costs and quality.

We prove that the socially optimal solution can be sustained as a market outcome by using a public firm as a market agent. We also provide conditions on the constant unit cost function under which every market outcome is a social optimum.

Duopole mixte dans un cadre de différentiation verticale

RÉSUMÉ. – Dans cet article, nous étudions un marché duopolistique avec produits différenciés verticalement. Le duopole est composé d'une firme privée (maximisant son profit) et d'une firme publique (maximisant le surplus total).

Les conditions technologiques sont supposées identiques pour les deux firmes. Ces conditions sont décrites par un coût unitaire constant par rapport à la quantité mais croissant par rapport à la qualité. Nous n'imposons pas de forme particulière à la relation entre coût unitaire et qualité.

Nous montrons que la présence d'une firme publique sur le marché permet à la solution socialement optimale d'être un équilibre de marché. Par ailleurs, nous montrons que moyennant certaines conditions suffisantes, tout équilibre de marché est un optimum social.

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1 Introduction

In oligopoly markets with quality choice, the market outcome does not generally coincide with the socially optimal solution (this result can be found in MOORTHY [1988] and is further developed in our section 2). Such a result opens the way for public intervention to recover optimality or, at least, to achieve some welfare improvement. One possible policy might be to propose a public intervention through market regulation in order to recover optimality. The aim of this paper is to present the mixed duopoly as a possible alternative to market regulation when the objective is to achieve first best outcomes. By mixed duopoly we mean a situation where a private firm, behaving as a profit maximizer, competes with a public firm that maximizes total surplus. The question we adress in this paper is whether, in a mixed duopoly, a socially optimal solution can be sustained as a market outcome by using a public firm as a market agent without giving the public firm any advantage over the private firm. We show that this is in fact the case and that it is the presence of a firm having total surplus as its objective and behaving strategically in the market that drives the result; no cost or order of move advantage over the other firm is necessary. Moreover, even though the socially optimal solution can be implemented through market regulation, the mixed duopoly alternative may offer some advantages when the informational requirements for an effective market regulation are taken into account. In fact, in a context of complete information, market regulation would lead to a social optimum; nevertheless it seems reasonable to argue that a public manager will easierly have information on demand and technological parameters than a regulation agency.

The analysis of mixed oligopolies has only recently been addressed in the economics literature. The first contributions in this domain focused on a homogeneous good market (for a survey on this literature up to 1989 see De FRAJA and DELBONO [1989]). However, most markets are in fact differentiated, and the need for the analysis of mixed oligopolies under product differentiation is apparent. To the best of our knowledge, the first attempt made in this direction is a paper by CREMER, MARCHAND and THISSE [1991] who analyze horizontal product differentiation in a Hotelling-type model.

The present paper studies a mixed duopoly where vertical differentiation is modeled using the Mussa-Rosen utility function and under the assumption of covered market. As in most mixed oligopoly models, the objective of the public firm is assumed to be the maximization of the total surplus under the constraint of non-negative profits while the private firm maximizes profit. The technology, identical for both firms, is described by unit costs which are constant with respect to quantity, though increasing in quality. No specific form is given to the relation between unit costs and quality. We consider a two stage game where, in the first stage, firms simultaneously select quality and, in the second stage, prices are simultaneously chosen. We look for subgame perfect pure strategy Nash equilibria.

Our main result is that, in the framework of this model, every social optimum can be sustained as a subgame perfect equilibrium whatever the constant unit cost function. We also provide conditions on the cost function under which the converse holds, *i. e.*, every subgame perfect equilibrium is a social optimum.

The remainder of the paper is organized as follows. In section 2 we present the model, characterize the social optimum and the private duopoly outcome and contrast them. Section 3 studies the price equilibrium for any given pair of qualities in the mixed duopoly. In section 4, the quality game in the mixed duopoly is analyzed. Subsection 4-a addresses the case of a strictly convex (in quality) unit cost function, and subsection 4-b deals with the case of a concave unit cost function. Section 5 is devoted to some final remarks.

2 Private Duopoly and Social Optimum

In this section we introduce the basic model and present the price and quality equilibrium of a private duopoly. We briefly discuss the social optimum and contrast it with the private duopoly outcome.

Consider the following model of vertical quality differentiation. There are two firms, each producing a good of quality Q_i $(i = 1,\, 2)$ and facing the same production technology which is characterized by constant costs per unit of output. The constant unit cost is increasing in quality.

The demand side is described by a population of heterogeneous consumers whose marginal willingness to pay for quality is uniformly distributed over the interval $[\underline{\theta},\, \overline{\theta}]$, with $\overline{\theta} > \underline{\theta} > 0$. Each consumer buys a single unit of one of the variants at the exclusion of the other. Consumer of type θ is characterized by the utility function $\mathrm{U}_\theta = \theta\,\mathrm{Q} - \mathrm{P}$ where P stands for the price charged for one unit of a good of quality Q (see, e. g. MUSSA and ROSEN [1978]).

In the first stage of the game firms simultaneously choose a level of quality, which cannot be changed afterwards. In the second stage, once qualities have been chosen and are known by both firms, firms simultaneously set their prices.

2.1. **Social Optimum**

We define the social optimum as the solution to the maximization of total surplus, where total surplus means the sum of the consumers' surplus and the profits accruing to the two firms in the market. In the present model the social optimum is defined as a pair of qualities and a splitting of consumers between these two qualities.

Let us first characterize the optimal splitting of consumers between any two levels of quality for pairs $(\mathrm{Q}_1,\, \mathrm{Q}_2)$ such that $\mathrm{Q}_1 > \mathrm{Q}_2$. The social cost of switching one consumer from consuming a good of lower quality (Q_2) to consuming a good of higher quality (Q_1) is given by $C(Q_1) - C(Q_2)$. On the order hand the social benefit accruing from switching consumer θ from quality Q_2 to quality Q_1 is mesured by $\theta(Q_1 - Q_2)$. Thus the optimal splitting of consumers requires that all consumers with $\theta < \dfrac{C(Q_1) - C(Q_2)}{Q_1 - Q_2}$ be assigned to the lower quality and all consumers with $\theta > \dfrac{C(Q_1) - C(Q_2)}{Q_1 - Q_2}$ be assigned to the higher quality. For notational simplicity we will hereafter refer to $\dfrac{C(Q_1) - C(Q_2)}{Q_1 - Q_2}$ as $\alpha(Q_1, Q_2)$ which can be read as the incremental cost per unit of quality when moving from quality Q_2 to quality Q_1.

In particular the optimal splitting rule means that whenever $\alpha(Q_1, Q_2) > \overline{\theta}$ all consumers should be assigned to the lower quality and when $\alpha(Q_1, Q_2) < \underline{\theta}$ all consumers should be assigned to the higher quality. Both levels of quality should be produced only if they are such that $\alpha(Q_1, Q_2) \in]\underline{\theta}, \overline{\theta}[$.

When the social optimum requires two qualities to be present on the market these qualities satisfy the following system of first order conditions for total surplus maximization:

$$-2\,C'(Q_1) + \alpha(Q_1, Q_2) + \overline{\theta} = 0 \tag{1}$$

$$-2\,C'(Q_2) + \alpha(Q_1, Q_2) + \underline{\theta} = 0 \tag{2}$$

2.2. **Private Duopoly**

We start by determining the price equilibrium for each pair of qualities (Q_1, Q_2). Let $Q_1 > Q_2$ and define the marginal consumer θ^* as the consumer who is indifferent between the two firms given the prices (P_1, P_2) and the qualities (Q_1, Q_2). Clearly $\theta^* = \dfrac{P_1 - P_2}{Q_1 - Q_2}$ and consumers of type $\theta > \theta^*$ patronize firm 1 (firm with the higher quality) while consumers of type $\theta < \theta^*$ buy from firm 2. By maximizing one firm's profit function with respect to its price, taking the other firm's price as a parameter, we get the best reply for each firm [1]. The best reply of firm 1 writes as

$$P_1 = P_2 + \underline{\theta}(Q_1 - Q_2) \quad \text{if} \quad P_2 \geq C(Q_1) + (\overline{\theta} - 2\,\underline{\theta})(Q_1 - Q_2) \tag{3}$$

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1. The profit function of firm 1 writes as:

$$\begin{cases} (P_1 - C(Q_1))\,(\overline{\theta} - \underline{\theta}) & \text{if} \quad P_1 \leq P_2 + \underline{\theta}(Q_1 - Q_2) \\ (P_1 - C(Q_1))\left(\overline{\theta} - \dfrac{P_1 - P_2}{Q_1 - Q_2}\right) & \text{if} \quad P_2 + \underline{\theta}(Q_1 - Q_2) \leq P_1 \leq P_2 + \overline{\theta}(Q_1 - Q_2) \\ 0 & \text{if} \quad P_1 \geq P_2 + \overline{\theta}(Q_1 - Q_2) \end{cases}$$

(4) $\mathrm{P}_1 = \dfrac{\mathrm{P}_2 + \overline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2) + \mathrm{C}\,(\mathrm{Q}_1)}{2}$

if $\mathrm{C}\,(\mathrm{Q}_1) - \overline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2) < \mathrm{P}_2 \leq \mathrm{C}\,(\mathrm{Q}_1) + \big(\overline{\theta} - 2\,\underline{\theta}\big)\,(\mathrm{Q}_1 - \mathrm{Q}_2)$

(5) $\mathrm{P}_1 \geq \mathrm{P}_2 + \overline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2)$ if $\mathrm{P}_2 \leq \mathrm{C}\,(\mathrm{Q}_1) - \overline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2)$

and the best reply of firm 2 writes as

(6) $\mathrm{P}_2 = \mathrm{P}_1 - \overline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2)$ if $\mathrm{P}_1 \geq \mathrm{C}\,(\mathrm{Q}_2) + \big(2\,\overline{\theta} - \underline{\theta}\big)\,(\mathrm{Q}_1 - \mathrm{Q}_2)$

(7) $\mathrm{P}_2 = \dfrac{\mathrm{P}_1 - \underline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2) + \mathrm{C}\,(\mathrm{Q}_2)}{2}$

if $\mathrm{C}\,(\mathrm{Q}_2) + \underline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2) < \mathrm{P}_1 \leq \mathrm{C}\,(\mathrm{Q}_2) + \big(2\,\overline{\theta} - \underline{\theta}\big)\,(\mathrm{Q}_1 - \mathrm{Q}_2)$

(8) $\mathrm{P}_2 \geq \mathrm{P}_1 - \underline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2)$ if $\mathrm{P}_1 \leq \mathrm{C}\,(\mathrm{Q}_2) + \underline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2)$

Combining these two best reply correspondences allows us to compute the following price equilibrium of a private duopoly:

When $\alpha\,(\mathrm{Q}_1,\,\mathrm{Q}_2) < 2\,\underline{\theta} - \overline{\theta}$, all pairs $(\overline{p}_1,\,\overline{p}_2)$ such that

(9) $$\begin{cases} \overline{p}_1 \in [\mathrm{C}\,(\mathrm{Q}_1) + \big(\overline{\theta} - \underline{\theta}\big)\,(\mathrm{Q}_1 - \mathrm{Q}_2),\ \mathrm{C}\,(\mathrm{Q}_2) + \underline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2)] \\ \text{and} \\ \overline{p}_1 = \overline{p}_2 + \underline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2) \end{cases}$$

When $\alpha\,(\mathrm{Q}_1,\,\mathrm{Q}_2) \in [2\,\underline{\theta} - \overline{\theta},\ 2\,\overline{\theta} - \underline{\theta}]$

(10) $$\begin{cases} \overline{p}_1 = \dfrac{\mathrm{C}\,(\mathrm{Q}_2) + 2\,\mathrm{C}\,(\mathrm{Q}_1) + \big(2\,\overline{\theta} - \underline{\theta}\big)\,(\mathrm{Q}_1 - \mathrm{Q}_2)}{3} \\ \overline{p}_2 = \dfrac{2\,\mathrm{C}\,(\mathrm{Q}_2) + \mathrm{C}\,(\mathrm{Q}_1) + \big(\overline{\theta} - 2\,\underline{\theta}\big)\,(\mathrm{Q}_1 - \mathrm{Q}_2)}{3} \end{cases}$$

When $\alpha\,(\mathrm{Q}_1,\,\mathrm{Q}_2) > 2\,\overline{\theta} - \underline{\theta}$, all pairs $(\overline{p}_1,\,\overline{p}_2)$ such that

(11) $$\begin{cases} \overline{p}_2 \in [\mathrm{C}\,(\mathrm{Q}_2) + \big(\overline{\theta} - \underline{\theta}\big)\,(\mathrm{Q}_1 - \mathrm{Q}_2),\ \mathrm{C}\,(\mathrm{Q}_1) + \overline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2)] \\ \text{and} \\ \overline{p}_2 = \overline{p}_1 - \overline{\theta}\,(\mathrm{Q}_1 - \mathrm{Q}_2) \end{cases}$$

The splitting of consumers between the two firms implied by this price equilibrium departs from the socially optimal splitting in the interval of values of $\alpha\,(\mathrm{Q}_1,\,\mathrm{Q}_2)$ for which both firms operate on the market. More precisely, under a private duopoly, the price equilibrium corresponding to pairs of quality for which $\alpha\,(\mathrm{Q}_1,\,\mathrm{Q}_2) \in [2\,\underline{\theta} - \overline{\theta},\ \underline{\theta}] \cup [\overline{\theta},\ 2\,\overline{\theta} - \underline{\theta}]$ leads to a market configuration where both firms are active. We have seen that for pairs of quality such that $\alpha\,(\mathrm{Q}_1,\,\mathrm{Q}_2) \notin [\underline{\theta},\,\overline{\theta}]$ the optimal allocation of consumers would require only one firm to be present on the market. Moreover, for the pairs of qualities for which it is socially optimal that both firms be present on the market, the allocation of consumers between the two qualities emerging from this price equilibrium is not optimal except for quality levels such that $\alpha(Q_1, Q_2) = \dfrac{\underline{\theta} + \overline{\theta}}{2}$. It then remains to study whether the quality equilibrium in the private duopoly is socially optimal. We will restrict our attention to the cases where the social optimum requires both firms to be present on the market.

An interior equilibrium satisfies the following system of first order conditions for profit maximization:

$$-2\,C'(Q_1) + \alpha(Q_1, Q_2) + 2\,\overline{\theta} - \underline{\theta} = 0 \tag{12}$$

$$-2\,C'(Q_2) + \alpha(Q_1, Q_2) + 2\,\underline{\theta} - \overline{\theta} = 0 \tag{13}$$

Recall that the socially optimal quality levels satisfy conditions (1) and (2). Clearly, a pair (Q_1, Q_2) cannot simultaneously satisfy the above system and the system formed by (1) and (2).

> PROPOSITION 1 : If the social optimum solution requires two different qualities to be present in the market, the social optimum cannot be sustained as the subgame perfect equilibrium outcome of a private duopoly.

Proposition 1 clearly indicates that, in general, the market outcome in a private duopoly is not socially optimal.

3 Price Equilibrium in the Mixed Duopoly

In this section we study the price equilibrium in a market where one firm is private (firm P), thus maximizing its profits, and the other firm is a public firm (firm S) which maximizes total surplus subject to a non-negative profit constraint.

In order to determine the price equilibrium in this market we start by caracterizing the objective function and the best reply of the public firm for the case where the public firm offers a higher quality than the private firm $(Q_S > Q_P)$.

Total surplus is given by

$$\begin{aligned} TS = CS + \Pi_P + \Pi_S = & \int_{\underline{\theta}}^{\theta^*} (\theta\, Q_P - C(Q_P))\, f(\theta)\, d\theta \\ & + \int_{\theta^*}^{\overline{\theta}} (\theta\, Q_S - C(Q_S))\, f(\theta)\, d\theta \end{aligned}$$

where CS stands for consumer surplus generated by both firms, Π_P for the private firm's prodit and Π_S for the public firm's profit. After some simple manipulations the total surplus can be written as:

$$\frac{1}{2\left(\overline{\theta}-\underline{\theta}\right)}\left[Q_P\left(\overline{\theta}^2-\underline{\theta}^2\right)-2\,C\left(Q_P\right)\left(\overline{\theta}-\underline{\theta}\right)\right]$$
$$\text{if}\quad P_S > P_P + \overline{\theta}\left(Q_S - Q_P\right)$$

$$\frac{1}{2\left(\overline{\theta}-\underline{\theta}\right)}\left[-\frac{\left(P_S-P_P\right)^2}{Q_S-Q_P}+Q_S\,\overline{\theta}^2-Q_P\,\underline{\theta}^2\right.$$
$$\left.+\,2\,\frac{P_S-P_P}{Q_S-Q_P}\left(C\left(Q_S\right)-C\left(Q_P\right)\right)+2\,C\left(Q_P\right)\underline{\theta}-2\,C\left(Q_S\right)\overline{\theta}\right]$$
$$\text{if}\quad P_P+\underline{\theta}\left(Q_S-Q_P\right)\leq P_S\leq P_P+\overline{\theta}\left(Q_S-Q_P\right)$$

$$\frac{1}{2\left(\overline{\theta}-\underline{\theta}\right)}\left[Q_S\left(\overline{\theta}^2-\underline{\theta}^2\right)-2\,C\left(Q_S\right)\left(\overline{\theta}-\underline{\theta}\right)\right]$$
$$\text{if}\quad P_S < P_P + \underline{\theta}\left(Q_S - Q_P\right)$$

The best reply of the public firm in the price game is then the price P_S that, for each given value of the private firm's price P_P, maximizes total surplus under the non-negative profit constraint. The public firm's best reply when $Q_S > Q_P$ is then given by

A) When $\dfrac{C\left(Q_S\right)-C\left(Q_P\right)}{Q_S-Q_P}\leq\underline{\theta}$

$$\begin{cases} C\left(Q_S\right)\leq P_S\leq P_P+\underline{\theta}\left(Q_S-Q_P\right) \\ \quad\text{if}\quad P_P\geq C\left(Q_S\right)-\underline{\theta}\left(Q_S-Q_P\right) \\ P_S=C\left(Q_S\right)\qquad\text{otherwise}\end{cases}\tag{14}$$

B) When $\underline{\theta}\leq\dfrac{C\left(Q_S\right)-C\left(Q_P\right)}{Q_S-Q_P}\leq\overline{\theta}$

$$\begin{cases} P_S=P_P+C\left(Q_S\right)-C\left(Q_P\right) \\ \quad\text{if}\quad P_P\geq C\left(Q_P\right) \\ P_S=C\left(Q_S\right)\qquad\text{otherwise}\end{cases}\tag{15}$$

C) When $\dfrac{C\left(Q_S\right)-C\left(Q_P\right)}{Q_S-Q_P}\geq\overline{\theta}$

$$P_S\geq P_P+\overline{\theta}\,\left(Q_S-Q_P\right)\tag{16}$$

The interpretation of this best reply correspondance is quite straightforward if we recall the properties of the socially optimal splitting of consumers between two qualities from the preceeding section. Let us start with (15). We have seen that, for qualities such that $\underline{\theta}\leq\alpha\left(Q_S,\,Q_P\right)\leq\overline{\theta}$, the optimal splitting requires both firms to be active in the market and the indifferent consumer, θ^*, to be given by $\alpha(Q_S, Q_P)$. The fact that the objective function of the public firm is the total surplus means that, from the point of view of the public firm, prices are just transfers between consumers and firms. In other words, to ensure the optimal splitting all that is needed is that the price difference between the two firms be given by $C(Q_S) - C(Q_P)$. Thus, by setting $P_S = P_P + C(Q_S) - C(Q_P)$ the public firm is able to enforce the optimal assignement of consumers between the two firms. In order to respect the non-negative profit constraint the public firm is forced to set $P_S = C(Q_S)$ whenever the previous reply implies a price below its unit cost, but since this would only happen for $P_P < C(Q_P)$, at equilibrium the budget constraint is not binding.

As for (14) we have seen that whenever $\alpha(Q_S, Q_P) < \underline{\theta}$ all consumers should optimally be assigned to the highest quality, in this case to the public firm. By setting any price such that $P_S \leq P_P + \underline{\theta}(Q_S - Q_P)$ the public firm can satisfy this splitting. The budget constraint implies that only prices equal or above $C(Q_S)$ can be chosen. So, depending on whether P_P is big enough or not, the public firm will be able or not to reply with a price that drives the private firm out of the market.

In a similar fashion, we have seen that for qualities such that $\alpha(Q_S, Q_P) > \overline{\theta}$ the optimal assignement of consumers requires that only the low quality firm (in this case the private firm) be active on the market. In this region the public firm should then set its price high enough to be driven out of the market. More precisely any price satisfying $P_S \geq P_P + \overline{\theta}(Q_S - Q_P)$ will entail the desired result. In this case the budget constraint is never binding since the reply of the public firm ensures that its market share is zero.

The computation of the payoff functions and of the best reply functions of both firms for the case $Q_P > Q_S$ is similar. The price equilibrium for any quality configuration where $Q_P \neq Q_S$ is given by [2]:

A) For $\dfrac{C(Q_S) - C(Q_P)}{Q_S - Q_P} < \underline{\theta}$,

$Q_S > Q_P$ (Region I)

all pairs $(\overline{P}_P, \overline{P}_S)$ such that:

$$C(Q_S) \leq \overline{P}_S \leq C(Q_P) + \underline{\theta}(Q_S - Q_P) \quad \text{and} \quad \overline{P}_S \leq \overline{P}_P + \underline{\theta}(Q_S - Q_P)$$

$Q_S < Q_P$ (Region II)

all pairs $(\overline{P}_P, \overline{P}_S)$ such that:

$$\overline{P}_S = \overline{P}_P - \underline{\theta}\,(Q_P - Q_S) \quad \text{and} \quad \overline{P}_S \geq C(Q_P) - (2\,\underline{\theta} - \overline{\theta})\,(Q_P - Q_S)$$

B) For $\underline{\theta} < \dfrac{C(Q_S) - C(Q_P)}{Q_S - Q_P} < \overline{\theta}$,

2. When $Q_P = Q_S$, $P_P = P_S = C(Q_P)$ is a price equilibrium in which both firms earn zero profits and total surplus reduces to consumer surplus. Moreover, this will be the unique price equilibrium if it is assumed that, with equal prices, both firms get some share of the market.

$Q_S > Q_P$ (Region III)

$$\overline{P}_S = 2\,C(Q_S) - C(Q_P) - \underline{\theta}\,(Q_S - Q_P)$$

and

$$\overline{P}_P = C(Q_S) - \underline{\theta}\,(Q_S - Q_P)$$

$Q_S < Q_P$ (Region IV)

$$\overline{P}_S = 2\,C(Q_S) - C(Q_P) - \overline{\theta}\,(Q_S - Q_P)$$

and

$$\overline{P}_P = C(Q_S) - \overline{\theta}\,(Q_S - Q_P)$$

C) For $\dfrac{C(Q_S) - C(Q_P)}{Q_S - Q_P} > \overline{\theta}$,

$Q_S > Q_P$ (Region V)

all pairs $(\overline{P}_P, \overline{P}_S)$ such that:

$$\overline{P}_S = \overline{P}_P + \overline{\theta}\,(Q_S - Q_P)$$

and

$$\overline{P}_S \geq C(Q_P) + (2\,\overline{\theta} - \underline{\theta})\,(Q_S - Q_P)$$

$Q_S < Q_P$ (Region VI)

all pairs $(\overline{P}_P, \overline{P}_S)$ such that:

$$C(Q_S) \leq \overline{P}_S \leq C\,(Q_P) - \overline{\theta}\,(Q_P - Q_S)$$

and

$$\overline{P}_S \leq \overline{P}_P - \overline{\theta}\,(Q_P - Q_S)$$

The most important feature of this price equilibrium configuration is that, *for any given pair of qualities, the allocation of consumers between qualities is socially optimal*. Indeed, we see that when $\alpha(Q_1, Q_2) \notin [\underline{\theta}, \overline{\theta}]$, the price equilibria are always such that the whole market is served by the firm selling the sole quality that would be chosen by all consumers if both qualities were to be offered at their marginal cost. Furthermore, when $\alpha(Q_1, Q_2) \in [\underline{\theta}, \overline{\theta}]$, the splitting of consumers is optimal since the price difference $(P_S - P_P)$ equals the marginal cost difference $(C(Q_S) - C(Q_P))$ so that $\theta^* = \alpha(Q_P, Q_S)$.

The fact that, at given qualities, the equilibrium prices lead to the optimal splitting of consumers depends crucially on two assumptions: (i) in the objective function of the public firm the consumer surplus and the producer surplus have the same weight and (ii) there is no quantity effect since all consumers are supposed to buy one unit either from firm P or from firm S. These two assumptions make that from a social point of view it is the price difference, rather than the absolute value of prices, that matters. Being so, the public firm by setting its choice variable (its price) is automatically able to set the desired price difference [3].

3. In fact, given the above two mentioned assumptions, the optimality of the price game outcome is independent of the utility function chosen and of the distribution of consumer's willingness to pay.

The above property implies that when the total surplus function is evaluated at the equilibrium prices, the values of Q_S and Q_P which maximize this function are the socially optimal qualities. Clearly the function $T_S(Q_P, Q_S)$, in which prices have been replaced by the equilibrium prices, is symmetric about the 45 degree line in the qualities space. The intuition behind this is that, from the social point of view, it is not important which firm produces which quality, as long as the allocation of consumers is optimal for every pair of qualities. As we have seen above, this is ensured by the price equilibria.

Notice that the only effect of the non-negative profit constraint is that within regions I and VI in the set of price equilibria, P_S is bounded from below by $C(Q_S)$. In the absence of this constraint the set of price equilibria would then be enlarged.

Whenever $\alpha \notin [\underline{\theta}, \overline{\theta}]$, that is in cases I, II, V, and VI, multiple price equilibria occur. In any of these cases the price equilibria lead to a situation where only one firm keeps the market. We have seen that, for the public firm, the only role of the prices is to assign consumers to the two qualities on the market, so in any of these cases total surplus is independent of the equilibrium prices. In cases I and VI, whatever the price equilibrium, the market share of the private firm is zero, so, all price equilibria are payoff equivalent. On the contrary, in cases II and V the private firm keeps the whole market, therefore its profit depends on the price equilibrium. In conclusion, the only cases in which the multiplicity of price equilibria might matter is in cases II and V, and only with regard to the objective function of the private firm.

4 Quality Equilibrium in the Mixed Duopoly

We are now in position to study the quality game. From the results of the price equilibrium we know that each possible pair (Q_P, Q_S) belongs to one of the six regions defined by the six cases mentioned above. For each pair (Q_P, Q_S) belonging to regions I, III, IV or VI the corresponding payoff function for both payers is well defined. This is so, since in regions III and IV, the price equilibrium is unique and in regions I and VI, as we have already mentioned in section 3, the price multiplicity does not influence the value of the payoff functions. On the contrary, within regions II and V the payoff of the private firm depends on the price equilibrium selected.

The payoff functions for both players corresponding to the six regions are presented in appendix A.

We now state two lemmas that will be used in the proof of Proposition 4. The first lemma applies when (Q_P, Q_S) belongs to region I, III, IV or VI while the second one applies when (Q_P, Q_S) belongs to regions II or V.

LEMMA 2 : For every pair (Q_P, Q_S) such the (i) $\alpha(Q_P, Q_S) \in [\underline{\theta}, \overline{\theta}]$, or (ii) $\alpha(Q_P, Q_S) < \underline{\theta}$ with $Q_S > Q_P$, or (iii) $\alpha(Q_P, Q_S) > \overline{\theta}$ with $Q_S < Q_P$ the private firm's profit can be written as $\Pi_P = 2(\overline{\theta} - \underline{\theta})\,TS + F(Q_S)$ where TS stands for total surplus and $F(Q_S) \equiv 2(\overline{\theta} - \underline{\theta})\,C(Q_S) - (\overline{\theta}^2 - \underline{\theta}^2)\,Q_S$.

Lemma 2 can be proven by simple manipulation of the payoff functions corresponding to the four regions to which it applies.

LEMMA 3 : There exists a price selection rule for which the relation between the private firm's profit and total surplus, for pairs (Q_P, Q_S) such that $\alpha(Q_P, Q_S) < \underline{\theta}$ with $Q_S < Q_P$ or $\alpha(Q_P, Q_S) > \overline{\theta}$ with $Q_S > Q_P$, is given by $\Pi_P = 2\left(\overline{\theta} - \underline{\theta}\right) TS + F(Q_S)$ where TS stands for total surplus and $F(Q_S) \equiv 2\left(\overline{\theta} - \underline{\theta}\right) C(Q_S) - (\overline{\theta}^2 - \underline{\theta}^2)\,Q_S$.

The proof of this lemma is given in appendix B.

PROPOSITION 4 : Any social optimum can be sustained as a subgame perfect equilibrium.

Proof: Let a pair (Q_P^{SO}, Q_S^{SO}) stand for a social optimum. Clearly, given Q_P^{SO}, Q_S^{SO} maximizes total surplus. Similarly, given Q_S^{SO}, by definition of a social optimum, Q_P^{SO} is the value of Q_P that maximizes TS. However, by Lemmas 2 and 3, if Q_P^{SO} maximizes TS it also maximizes Π. Hence, (Q_P^{SO}, Q_S^{SO}) is an equilibrium of the first stage game for the price selection rule presented above [4].

One might wonder under which circumstances the converse of proposition 4 does hold, that is, under which assumptions can we say that every subgame perfect equilibrium is a social optimum. To study this question we have to keep in mind that subgame perfect equilibria that are not social optima may arise from two different reasons. First, the multiplicity of price equilibria and consequently of the private firm's payoff within regions II and V may generate some non-socially optimal equilibria. Second, even when the price multiplicity is not an issue, that is, even if whatever the price equilibrium selection no quality equilibrium exists within regions II or V, there may still exist some quality equilibria that are not socially optimal.

To tackle this question we will study separately the case of a convex and a concave unit cost function. It is only when the unit cost function does not present inversions of concavity that the six regions can be given a simple graphical representation in the space (Q_S, Q_P). This representation proves helpful in showing that, when the unit cost function is all over concave

4. LEDERER and HURTER [1986] studied a two stage private duopoly model where firms decide first on their location and then on price schedules defined over a market space. They show in this model that the profit of a firm in the location game can be written as the difference between a function of the strategic variable of the other firm and the social cost. This result is somehow equivalent to the one presented in Lemmas 2 and 3.

or all over convex, the multiplicity of price equilibria in regions II and V does not constitute a problem.

For the assumption of a concave unit cost to be meaningful, we must assume that the interval of feasible qualities is bounded. To perform the study of the two cases (convex and concave unit cost function) under the same assumptions we will assume in the two following subsections that qualities have to be chosen from the interval $[\underline{Q}, \overline{Q}]$ where $\underline{Q} \geq 0$.

Moreover, there is a main difference between the cases of strictly convex and concave unit cost functions. In the case of a strictly convex function, the fact that only two qualities are present in the market constitutes a constraint in terms of achieving maximum welfare. In other words, if it were possible to produce any number of products, the social optimum solution would lead to a continuum of qualities [5]. The intuition behind this result is that if each consumer was asked to choose a level of quality, while paying the marginal cost, each consumer would choose a different level (in particular a consumer of type θ would choose a level of quality such that $C'(Q) = \theta$). On the contrary, whenever the unit cost function is concave, the optimal number of qualities is at most two.

4.1. **The Convex Unit Cost Function**

In this subsection and the next we further assume that the unit cost function is a continuous twice differentiable function. The assumption of differentiability is not crucial but significantly simplifies the proofs.

Figure 1 depicts the regions of quality pairs corresponding to each of the six possible price equilibria. Notice that when $Q_P \rightarrow Q_S$, $\alpha \rightarrow C'(Q_S)$. The curved lines represent the pairs of qualities for which $\alpha(Q_P, Q_S) = \underline{\theta}$ and $\alpha(Q_P, Q_S) = \overline{\theta}$. $Q_S^{max} = Q_P^{max}$ is the value of Q for which $C'(Q) = \overline{\theta}$; and $Q_S^{min} = Q_P^{min}$is the value of Q for which $C'(Q) = \underline{\theta}$. Figure 1 was drawn under the assumption that pairs (Q_P, Q_S) for which an interior price equilibrium exists are technologically feasible. This requires that $]Q^{min}, Q^{max}[\cap]\underline{Q}, \overline{Q}[\neq \emptyset$, which amounts to requiring that at least some consumers, when facing a price equal to marginal cost, would choose a quality level belonging to $]\underline{Q}, \overline{Q}[$.

> LEMMA 5 : Assume that interior market solutions are technologically feasible. Then, given a convex unit cost function, if an equilibrium in qualities exists it will be interior, *i.e.*, $\alpha(Q_P^*, Q_S^*) \in]\underline{\theta}, \overline{\theta}[$ whatever the price selection rule.

The proof of Lemma 5 is given in Appendix B.

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5. More precisely the support of this continuum of qualities would be $[\max(\underline{Q}, Q^{min}), \min(\overline{Q}, Q^{max})]$, where Q^{min} and Q^{max} are such that $C'(Q^{min}) = \underline{\theta}$ and $C'(Q^{max}) = \overline{\theta}$.

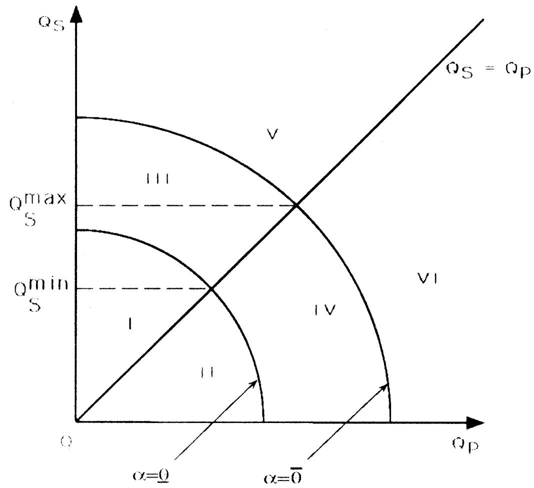

FIGURE 1

From the preceding section we know that an equilibrium in qualities exists. Furthermore, Lemma 5 ensures that all quality equilibria are strictly interior. Hence, quality equilibria which are not socially optimal can only arise if the total surplus function within region III has several separate maxima (maxima in each variable separately) yielding different values for which both coordinates are different. Note that since the total surplus function within regions IV and III is symmetric about the diagonal, we can concentrate on one of these two regions. Imposing quasi-concavity of the total surplus function within region III is a sufficient, but not necessary, condition for ensuring that every subgame perfect equilibrium is a social optimum (when the unit cost function is convex). Observe that Proposition 4, together with Lemma 5, implies that if interior market solutions are technologically feasible and the unit cost function is convex, then the socially optimal solution requires two different qualities.

In Appendix C we study the special case of a unit cost described by the convex and increasing branch of a cubic function. We show that, for this class of functions, the quality equilibrium is unique, which implies that it coincides with the social optimum.

What happens if $]Q^{min}, Q^{max}[\cap]\underline{Q}, \overline{Q}[=\varnothing$? This means that either $\underline{Q} > Q^{max}$, in which case all consumers would choose $\underline{Q}$ when any potential quality is offered at marginal cost, or $\overline{Q} < Q^{min}$, in which case all consumers would choose $\overline{Q}$. If any of these two inequalities holds, then the quality equilibrium will always be a social optimum since the public firm can always set $Q_S = Q^*$ (Q^* being either $\underline{Q}$ or $\overline{Q}$) and drive the private firm out of the market. Alternatively, if the private firm chooses $Q_P = Q^*$, the public firm will reply indifferently with a $Q_S \in [\underline{Q}, \overline{Q}]$. Hence, if the unit cost function is convex and $]Q^{min}, Q^{max}[\cap]\underline{Q}, \overline{Q}[=\varnothing$, every subgame perfect equilibrium is a social optimum (and of course vice-versa by the argument in the preceding section).

4.2. **The Concave Unit Cost Function**

As we have already noticed, whenever the unit cost function is concave the optimal number of qualities is at most two. Indeed, if each quality was priced at its marginal cost, each consumer would choose either $\underline{Q}$ or $\overline{Q}$ [consumers with $\theta < \alpha\left(\underline{Q}, \overline{Q}\right)$ (resp. $\theta > \alpha\left(\underline{Q}, \overline{Q}\right)$) would chose $\underline{Q}$ (resp. $\overline{Q}$)]. In particular, if $\alpha\left(\underline{Q}, \overline{Q}\right) < \underline{\theta}$ or $\alpha\left(\underline{Q}, \overline{Q}\right) > \overline{\theta}$, the social optimum solution is such that only one quality is sold in the market. More precisely, when $\alpha\left(\underline{Q}, \overline{Q}\right) < \underline{\theta}$, all consumers would choose $\overline{Q}$; and when $\alpha\left(\underline{Q}, \overline{Q}\right) > \overline{\theta}$, all consumers would choose $\underline{Q}$. Clearly, in these two cases, the locus of social optima coincides with the best reply function of the public firm. In fact, if the private firm sets $Q_P \neq Q^*$ (where Q^* is either $\underline{Q}$ or $\overline{Q}$) the public firm can always reply with $Q_S = Q^*$, and if the private firm sets $Q_P = Q^*$ the public firm will reply indifferently with any $Q_S \in [\underline{Q}, \overline{Q}]$. Therefore, in these two cases, every subgame perfect equilibrium is a social optimum. If, on the other hand, $\alpha\left(\underline{Q}, \overline{Q}\right) \in [\underline{\theta}, \overline{\theta}]$, the optimal number of qualities is two, the optimal levels of quality being $\underline{Q}$ and $\overline{Q}$. Consumers will be optimally assigned between these qualities when the marginal consumer is of type $\theta = \alpha\left(\underline{Q}, \overline{Q}\right)$. Figure 2 presents the feasible regions in the quality space.

From the results in section 4, we know that points A and B are quality equilibria. To prove that every subgame perfect equilibria is a social optimum we need some intermediate results.

LEMMA 6 : Given a concave unit cost function, whenever $\alpha\left(\underline{Q}, \overline{Q}\right) \in [\underline{\theta}, \overline{\theta}]$, every quality equilibrium will be such that $\alpha\left(Q_P^*, Q_S^*\right) \in]\underline{\theta}, \overline{\theta}[$ whatever the price selection rule.

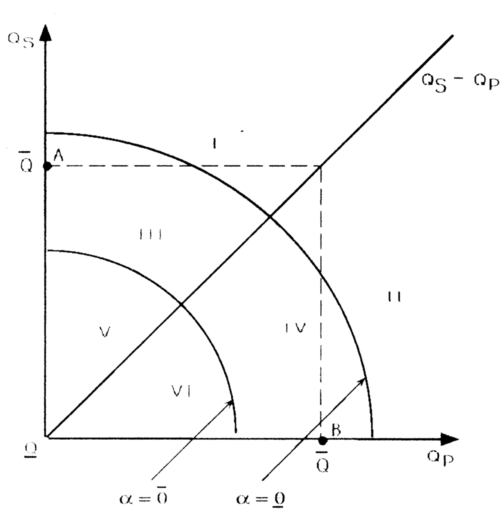

FIGURE 2

LEMMA 7 : Given a concave unit cost function, whenever $\alpha\left(\underline{Q}, \overline{Q}\right) \in [\underline{\theta}, \overline{\theta}]$, the only possible best reply of the public firm strictly within region III (resp. within region IV) are the points on the line $Q_S = \overline{Q}$ (resp. $Q_S = \underline{Q}$).

The proof of these two lemmas is presented in Appendix B.

Lemma 6 together with Lemma 7 implies that the only candidates to a subgame perfect equilibrium are the points such that $Q_P < Q_S = \overline{Q}$ and $\alpha\left(Q_P, \overline{Q}\right) > \underline{\theta}$ and the points such that $Q_P > Q_S = \underline{Q}$ and $\alpha\left(Q_P, \underline{Q}\right) < \overline{\theta}$. The study of the private firm's profit function shows that, within region III, this profit is decreasing in Q_P while within region IV the profit is increasing with Q_P. Hence, the only points that are simultaneously the best reply of the private firm and the best reply of the public firm are points A and B. Hence we have shown:

PROPOSITION 8 : When the unit cost function is concave, every perfect equilibrium is a social optimum.

5 Final Remarks

We have shown that, under the assumptions of a standard the model of vertical differentiation, the use of a public firm as a market agent which is not given any advantage over the private firm can lead to a socially optimal solution. In particular this means that the presence of a private firm does not necessarily disturb the optimality of the market outcome. Clearly, one has to bear in mind, specially when policy making is at stake, that the validity of the results has to be judged in the light of the assumptions made. In particular, assuming that the market is covered implies that this model is the more realistic the more it applies to markets with rather inelastic total demand. Some fields of health care and education as well as some durables' markets can be viewed as markets with total demand presenting some degree of inelasticity.

The results in this paper can be used in a slightly different context. The public firm could be viewed as a more regulated firm to which, in the limit, the quality level and price are imposed by a regulatory agency. If this is the case, then the strategic choices of the private firm, in response to the regulated variables of the other firm, will be socially optimal. In this case, the problem of multiple Nash equilibria in the quality game is avoided and the market outcome becomes necessarily a social optimum whatever the configuration of the unit cost function.

Objective functions

The expression of the objective functions of the two firms for the first stage game are the following:

Region	Private firm (Π)	Public firm (TS)
I	0	$\left(\overline{\theta}+\underline{\theta}\right)\frac{Q_S}{2}-C\left(Q_S\right)$
II	*indeterminate*	$\left(\overline{\theta}+\underline{\theta}\right)\frac{Q_P}{2}-C\left(Q_P\right)$
III	$\left(\alpha-\underline{\theta}\right)^2\left(Q_S-Q_P\right)$	$\frac{1}{2\left(\overline{\theta}-\underline{\theta}\right)}\left[-Q_P\,\underline{\theta}^2+Q_S\,\overline{\theta}^2+C\left(Q_S\right)\left(\alpha-2\,\overline{\theta}\right)+C\left(Q_P\right)\left(2\,\underline{\theta}-\alpha\right)\right]$
IV	$\left(\overline{\theta}-\alpha\right)^2\left(Q_P-Q_S\right)$	$\frac{1}{2\left(\overline{\theta}-\underline{\theta}\right)}\left[-Q_S\,\underline{\theta}^2+Q_P\,\overline{\theta}^2+C\left(Q_P\right)\left(\alpha-2\,\overline{\theta}\right)+C\left(Q_S\right)\left(2\,\underline{\theta}-\alpha\right)\right]$
V	*indeterminate*	$\left(\overline{\theta}+\underline{\theta}\right)\frac{Q_P}{2}-C\left(Q_P\right)$
VI	0	$\left(\overline{\theta}+\underline{\theta}\right)\frac{Q_S}{2}-C\left(Q_S\right)$

Proofs of Lemmas 3, 5, 6, 7

Proof of Lemma 3

$$\mathrm{TS}(\mathrm{V}) = \mathrm{TS}(\mathrm{II}) = \left(\overline{\theta} + \underline{\theta}\right) \frac{\mathrm{Q_P}}{2} - \mathrm{C}(\mathrm{Q_P})$$

$$\Pi(\mathrm{V}) = \Pi(\mathrm{II}) = \mathrm{P_P} - \mathrm{C}(\mathrm{Q_P})$$

For the equality $\Pi = 2\,\mathrm{TS} + \mathrm{F}(\mathrm{Q_S})$ to be verified the private firm's price has to be given by:

$$\mathrm{P_P^*} = \mathrm{C}(\mathrm{Q_P}) + (\mathrm{Q_S} - \mathrm{Q_P})\left(2\,\alpha - \underline{\theta} - \overline{\theta}\right)$$

Within region II any pair $(\mathrm{P_P},\, \mathrm{P_S})$ such that

$$\mathrm{P_S} > \mathrm{C}(\mathrm{Q_P}) - \left(2\,\underline{\theta} - \overline{\theta}\right)(\mathrm{Q_P} - \mathrm{Q_S}) \qquad \text{and} \quad \mathrm{P_S} = \mathrm{P_P} - \underline{\theta}\,(\mathrm{Q_P} - \mathrm{Q_S})$$

is a price equilibrium. These two conditions imply

$$\mathrm{P_P} > \mathrm{C}(\mathrm{Q_P}) + \left(\overline{\theta} - \underline{\theta}\right)(\mathrm{Q_P} - \mathrm{Q_S}).$$

This inequality is satisfied by the price selection rule $\mathrm{P_P^*}(\mathrm{Q_P},\, \mathrm{Q_S})$ for any pair $(\mathrm{Q_P},\, \mathrm{Q_S})$ such that $\mathrm{Q_P} > \mathrm{Q_S}$ and $\alpha(\mathrm{Q_P},\, \mathrm{Q_S}) < \underline{\theta}$. Hence for every point within region II, the proposed price selection rule corresponds to a price equilibrium. Similarly, within region V, the conditions for a pair $(\mathrm{P_P},\, \mathrm{P_S})$ to be an equilibrium imply $\mathrm{P_P} > \mathrm{C}(\mathrm{Q_P}) + \left(\overline{\theta} - \underline{\theta}\right)(\mathrm{Q_S} - \mathrm{Q_P})$. Once again, for all points in region V, this inequality is respected by the price selection rule.

Proof of Lemma 5

Assume that interior solutions are technologically feasible. Then:

a) *Elimination of regions I and VI*

Points in region I or VI will never be chosen by the private firm since $\Pi(\mathrm{I}) = \Pi(\mathrm{VI}) = 0$. Whatever the level of $\mathrm{Q_S}$, the private firm will never reply with a $\mathrm{Q_P} > \mathrm{Q_S}$ such that $\alpha(\mathrm{Q_P},\, \mathrm{Q_S}) \geq \overline{\theta}$. This excludes regions I and VI, as well as their boundaries, as possible locus of quality equilibria.

b) *Elimination of region V*

Given a convex unit cost function:

1 – For $\mathrm{Q_P} < \mathrm{Q^{max}}$, where $\mathrm{Q^{max}}$ is such that $\mathrm{C}'(\mathrm{Q^{max}}) = \overline{\theta}$, we have that:

$$\frac{\partial \mathrm{TS}(\mathrm{V})}{\partial \mathrm{Q_S}} = 0, \quad \frac{\partial \mathrm{TS}(\mathrm{III})}{\partial \mathrm{Q_S}}\Big|_{\alpha = \overline{\theta}} = 0 \quad \text{and} \quad \frac{\partial^2 \mathrm{TS}(\mathrm{III})}{\partial \mathrm{Q_S^2}}\Big|_{\alpha = \overline{\theta}} > 0$$

This means that for each $\mathrm{Q_P} < \mathrm{Q^{max}}$, there is a reply of the public firm strictly within region III which is better than any reply within region V. This excludes the upper part of the line $\alpha = \overline{\theta}$ and the subregion of region V to the left of $\mathrm{Q_P} = \mathrm{Q^{max}}$.

$2-$ For $Q_P > Q^{max}$

$\dfrac{\partial TS(V)}{\partial Q_S} = 0$ and the total surplus function is continuous along the diagonal.

Furthermore, $\dfrac{\partial TS(VI)}{\partial Q_S}|_{Q_S=\hat{Q}} = 0$ and $\dfrac{\partial^2 TS(VI)}{\partial Q_S^2}|_{Q_S=\hat{Q}} < 0$,

where $\hat{Q}$ is such that $C'(\hat{Q}) = \dfrac{\overline{\theta}+\underline{\theta}}{2}$, which implies that $\hat{Q} \in]Q^{min}, Q^{max}[$. This means that for $Q_P > Q^{max}$ there is a reply of the public firm strictly within region VI that is better than any reply within region V. This excludes the subregion of region V to the right of $Q_P = Q^{max}$.

$3-$ For $Q_P = Q^{max}$

The public firm has to choose a reply either within region IV or within region V.

$$\frac{\partial TS(V)}{\partial Q_S} = 0 \qquad \frac{\partial TS(IV)}{\partial Q_S}|_{Q_S=Q_P=Q^{max}} < 0$$

So, for $Q = Q^{max}$ a reply of the public firm strictly within region IV is better than any reply within region V.

Region II and its boundary can be eliminated from the locus of quality equilibria using the same kind of argument as for region V.

Since the elimination of regions II and V is based on the fact that the best reply function of the public firm lies outside those regions, and since these are the only regions where the price multiplicity matters, Lemma 3 is valid whatever the price equilibrium selection.

Proof of Lemma 6

$$\Pi(I) = \Pi(VI) = 0; \qquad \Pi(III) > 0; \qquad \Pi(IV) > 0$$

So, for every Q_S there is always a reply of the private firm strictly within region III (resp. region IV) that dominates a reply within region I (resp. region VI). This excludes regions I and VI, as well as their boundaries, from the locus of quality equilibria.

Given a concave unit cost function:

$$\frac{\partial TS(V)}{\partial Q_S} = 0, \quad \frac{\partial TS(III)}{\partial Q_S} > 0 \quad \text{and, for } \alpha = \overline{\theta}, \quad TS(V) = TS(III)$$

This means that any reply of the public firm within region III dominates any reply within V. In particular, this result is independent of the price selection rule since the value of the total surplus function is not affected by the price equilibrium selected. Hence, no quality equilibrium exists within region V or along its boundary.

$$\frac{\partial TS(II)}{\partial Q_S} = 0, \quad \frac{\partial TS(IV)}{\partial Q_S} > 0 \quad \text{and, for } \alpha = \underline{\theta}, \quad TS(II) = TS(IV).$$

The same reasoning as before excludes region II and its boundary from the locus of quality equilibria.

Proof of Lemma 7

This lemma is a direct consequence of the previous one. The fact that $\frac{\partial \mathrm{TS}(\mathrm{III})}{\partial \mathrm{Q_S}} > 0$ means that the only candidates to a best reply of the public firm, belonging to region III, are the points such that $\mathrm{Q_S} = \overline{\mathrm{Q}}$. Similarly, the fact that $\frac{\partial \mathrm{TS}(\mathrm{IV})}{\partial \mathrm{Q_S}} < 0$ implies that the best replies of the public firm, which strictly belong to region IV, are such that $\mathrm{Q_S} = \underline{\mathrm{Q}}$.

Cubic Unit Cost Function

In this appendix we show that if the unit cost function is described by the increasing convex branch of a cubic function the quality Nash equilibrium is unique within each interior region.

Given the symmetry properties between region III and IV we will concentrate on one of them-say region III.

Let $\mathrm{C}(\mathrm{Q}) = a\,\mathrm{Q} + \frac{b}{2}\mathrm{Q}^2 + \frac{c}{3}\mathrm{Q}^3 + d \qquad \text{for} \quad \mathrm{Q} \geq 0$
with $a,\ b,\ d \geq 0$ and $c > 0$.

Without loss of generality we set the lowest bound of the technologically feasible interval to be zero $\left(\underline{\mathrm{Q}} = 0\right)$. So, $\mathrm{C}\left(\underline{\mathrm{Q}}\right) = d$ and $\mathrm{C}'\left(\underline{\mathrm{Q}}\right) = a$. Technological feasibility of interior solutions implies $a < \overline{\theta}$.

We assume, for the sake of simplicity, that $\overline{\mathrm{Q}}$ is big enough, in particular we assume that the average cost $\mathrm{AC}\left(\overline{\mathrm{Q}}\right) > \overline{\theta}$, but it can be shown that the uniqueness of the quality equilibrium holds independently of this assumption.

Define the critical set of the best reply of the private firm belonging to region III as the locus of points $(\mathrm{Q_P},\ \mathrm{Q_S})$ such that:

$$\alpha\,(\mathrm{Q_P},\ \mathrm{Q_S}) \in [\underline{\theta},\ \overline{\theta}]; \qquad \mathrm{Q_S} \geq \mathrm{Q_P} \quad \text{and} \quad \frac{d\Pi}{d\mathrm{Q_P}} = 0$$

By eliminating at least some of the points for which $\frac{d\Pi}{d\mathrm{Q_P}} = 0$ but $\frac{d^2\Pi}{d\mathrm{Q_P^2}} > 0$ we can further reduce this set. More precisely, for any twice differentiable unit cost function we can write:

$$\frac{d\Pi}{d\mathrm{Q_P}} = (\alpha - \underline{\theta})\,(\alpha + \underline{\theta} - 2\,\mathrm{C}'\,(\mathrm{Q_P}))$$

and

$$\frac{d^2\,\Pi}{d\mathrm{Q_P^2}} = -2\,\mathrm{C}''(\mathrm{Q_P})\,(\alpha - \underline{\theta}) + 2\,\frac{(\alpha - \mathrm{C}'\,(\mathrm{Q_P}))^2}{\mathrm{Q_S} - \mathrm{Q_P}}$$

So, points $(\mathrm{Q_P},\ \mathrm{Q_S})$ such that $\alpha\,(\mathrm{Q_P},\ \mathrm{Q_S}) = \underline{\theta}$ can be eliminated. We define then the reduced critical set of the private firm's best reply belonging to region III as the locus of points $(\mathrm{Q_P},\ \mathrm{Q_S})$ such that:

$$\alpha\,(\mathrm{Q_P},\ \mathrm{Q_S}) \in\,]\underline{\theta},\ \overline{\theta}]; \qquad \mathrm{Q_S} \geq \mathrm{Q_P} \quad \text{and} \quad \alpha + \underline{\theta} - 2\,\mathrm{C}'\,(\mathrm{Q_P}) = 0$$

Clearly the best reply points (within region III) belong either to this reduced critical set or to $\alpha\,(\mathrm{Q_P},\ \mathrm{Q_S}) = \overline{\theta}$. Nevertheless we know from Proposition 4 and from Lemma 5 that Nash equilibria exist and are strictly interior, so, to study equilibria, we can ignore points on $\alpha\,(\mathrm{Q_P},\ \mathrm{Q_S}) = \overline{\theta}$ even if they belong to the private firm's best reply.

Note that point A in Figure 3 belongs to the reduced critical set since for $Q_P = Q_S = Q^{min}$ we have that $C'(Q_P) = \alpha = \underline{\theta}$ (which means that $\dfrac{d\Pi}{dQ_P} = 0$, though $\dfrac{d^2\Pi}{dQ_P^2} = 0$). Furthermore, there is a single point in $\alpha(Q_P, Q_S) = \overline{\theta}$ (point C in Figure 3) belonging to the reduced critical set, this point is such that $C'(Q_P) = \dfrac{\overline{\theta} + \underline{\theta}}{2}$.

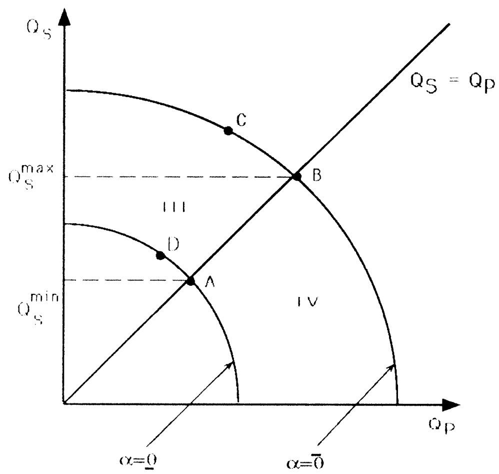

FIGURE 3

Exactly the same procedure allows us to define a reduced critical set of the public firm's best reply belonging to region III. This reduced critical set is given by the locus of points such that:

$$\alpha(Q_P, Q_S) \in [\underline{\theta}, \overline{\theta}[, \qquad Q_S \geq Q_P \quad \text{and} \quad \alpha + \overline{\theta} - 2\,C'(Q_S) = 0$$

Point B belongs to the critical set since for $Q_P = Q_S = Q^{max}$ we have that $C'(Q_S) = \alpha = \overline{\theta}$ $\left(\text{which means that } \dfrac{dTS}{dQ_S} = 0, \text{ though } \dfrac{d^2TS}{dQ_S^2} = 0\right)$. Point D, for which $\alpha(Q_P, Q_S) = \underline{\theta}$ and $C'(Q_S) = \dfrac{\overline{\theta} + \underline{\theta}}{2}$ is the only point on $\alpha(Q_P, Q_S) = \underline{\theta}$ belonging to the critical set.

We prove the uniqueness of the equilibrium within region III by showing that:

a) the reduced critical set of the private firm's best reply in region III is the graph of an increasing concave function Q_S of Q_P, going from point A to point C.

b) the reduced critical set of the public firm's best reply in region III is an increasing convex function going from point D to point B.

Given a) and b) these two critical sets functions cross only once and the crossing point is the unique Nash equilibrium.

Critical set of the private firm's best reply

A tedious computation shows that the points in the critical set within region III are the solutions, within this region, of:

$$Q_P = \frac{-9\,b + 2\,c\,Q_S + \sqrt{\Delta_1}}{20\,c}$$

with $\Delta_1 = 81\,b^2 + 84\,c^2\,Q_S^2 + 84\,bc\,Q_S + 240\,(\underline{\theta} - a)$

Clearly $\dfrac{dQ_P\,(Q_S)}{dQ_S} > 0$ and straightforward computation shows that $\dfrac{d^2Q_P\,(Q_S)}{dQ_S^2} > 0.$

So, the corresponding inverse function is an increasing concave function.

Critical set of the public firm's best reply

The points in the critical set within region III are the solutions, within this region, of:

$$Q_S = \frac{2\,c\,Q_P - 9\,b + \sqrt{\Delta_2}}{20\,c}$$

with $\Delta_2 = 81\,b^2 + 84\,c^2\,Q_P^2 + 84\,bc\,Q_P + 240\left(\overline{\theta} - a\right).$

Clearly $\dfrac{dQ_S\,(Q_P)}{dQ_P} > 0$ and straightforward computation shows that $\dfrac{d^2Q_S\,(Q_P)}{dQ_P^2} > 0.$

So, the critical set of the public firm's best reply is an increasing convex function.

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