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# Exchange Rate Target Zones in a Utility Maximizing Framework

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**ABSTRACT.** – We present a monetary model of unilateral exchange rate target zone based on microeconomic optimization principles. We endogenously derive the consumer's consumption and portfolio choices, and the demand for money balances. The standard monetary model results, like the S-shaped solution for the exchange rate and the transfer of volatility from exchange rates to nominal interest rates are also obtained.

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## Les zones cibles avec maximisation d'utilité

**RÉSUMÉ.** – Nous présentons un modèle monétaire de taux de change en zone cible unilatérale fondé sur des principes d'optimisation micro-économique. Nous dérivons de façon endogène la consommation et les choix de portefeuille du consommateur et la demande des encaisses monétaires. Les résultats habituels du modèle monétaire tels que la courbe en S du taux de change et le transfert de la volatilité des taux de change aux taux d'intérêt nominaux sont également obtenus.

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# 1 Introduction

This paper studies the dynamics of exchange rates and interest rates in a target zone regime. Contrary to the approach of the standard literature that is based on ad hoc monetary or sticky price models, ours is an optimizing approach.

We present a stochastic optimizing model of a small open pure exchange economy populated by a large number of risk averse utility maximizing individuals. We endogenously derive optimal consumption and portfolio choices and thus we get the demand for money, at least for the case of logarithmic utility, from an optimization process. Consumption is linear in wealth and independent of the exchange rate.

The paper is organized as follows: section 2 presents our model, section 3 solves for equilibrium consumption, portfolio choice, and the exchange rate under a target zone with enough reserves. Section 4 concludes.

# 2 The Model

We assume a small open economy with perfect capital mobility between itself and the rest of the world. Nominal money supply M equals the sum of domestic credit D and international reserves R, which consist of interest earning foreign bonds.

Money supply is assumed to equal the monetary base (high powered money). This way we abstract from the money creation process of the domestic banking system. Thus:

$$M = D + R, \quad R = Sr$$

where S stands for the nominal bilateral exchange rate, defined as the domestic currency price of one unit of the foreign currency. Reserves are valued at the original purchase exchange rate, and so the central bank is assumed not to monetize changes in the exchange rate.

The money supply is the only exogenous source of uncertainty in the economy and is assumed to follow a *regulated geometric Brownian Motion* process. A Brownian motion is the continuous time analogue of a random walk. Thus:

$$\text{(1)} \qquad \frac{dM}{M} = \mu\, dt + \sigma\, dz + dL - dU$$

$\mu$, $\sigma$ are positive constants, and $M(t_0)=M_0$ is given.

L and U are increasing, continuous and positive functions of the money supply.

$L = L(M)$, $U = U(M)$ with: $L_0 = U_0 = 0$.

L increases when $S = \underline{S}$, and U increases when $S = \overline{S}$ [1].

L and U are the total amounts of interventions at each boundary.

The quantity $\mu M$ is the "drift" of the process and $\sigma^2$, $M^2$, is the "diffusion".

$dz$ is a Wiener process which is the limit of a standard normal random variable, uncorrelated through time, with the following two properties:

$$E(dz) = 0$$

$$E(dz)^2 = dt.$$

E is the expectations operator conditional on information at time $t$. Information at time $t$ includes the fundamentals, money supply and foreign reserves, and any announcements the central bank has made concerning the exchange rate regime. Specifically it is assumed that the monetary autority has announced that it will defend the target zone as long as it has a stock of sufficient foreign reserves. Thus the band is perfectly credible while foreign reserves are above a certain critical level.

(1) in fact (without the regulators) is shorthand for the following integral equation:

$$M_t = M_0 + \int_0^t \mu M \, ds + \int_0^t \sigma M \zeta \, ds, \qquad 0 \leqq t \leqq T.$$

Where $\zeta$ is a white noise term equal by definition to the time derivative of a Wiener process. Thus:

$$\zeta = \frac{dz}{ds}$$

Taking expectations in (1) we have:

$$E\left(\frac{dM}{M}\right) = \mu \, dt$$

and so $\mu$ is the expected growth rate of money supply.

The money supply is regulated so that the exchange rate's oscillation is limited. This exchange rate regime is a managed float regime called a target zone. The exchange rate is controled – through foreign exchange non-sterilized intervention by central bank – between two barriers $\underline{S}$ and $\overline{S}$ with $\overline{S} > \underline{S}$, and $\underline{S}$, $\overline{S} > 0$. These interventions are infinitesimal. This way money supply is bounded between $\underline{M}$ and $\overline{M}$, The bounds on the fundamental with:

$$\underline{M}, \overline{M} > 0 \qquad \text{and} \qquad \overline{M} > \underline{M}$$

$\underline{M}$ is reached when $S = \underline{S}$, and $\overline{M}$ when $S = \overline{S}$.

1. For details on the theory of Regulated Brownian Motion see HARRISON M. [1985].

There is only one consumption good in the economy, therefore arbitrage will guarantee:

$$P = SP^* \quad \text{(The law of one price)}$$

The foreign price level P* is assumed to equal unity making the domestic price level equal to the exchange rate:

$$P = S$$

Given (1) we postulate that the domestic price level process takes the following form:

$$\text{(2)} \qquad \frac{dP}{P}\left(=\frac{dS}{S}\right) = \mu_P(M)\,dt + \sigma_p(M)\,dz$$

with: $\underline{S} < S < \overline{S}$.

Later on we prove that this is in fact the correct form for the price process. The above formulation captures the stylized fact that the logarithm of the exchange rate follows a random walk [2]. Actually (2) is a more general stochastic process than a random walk because as we will see later on $\mu_P$ and $\sigma_P$ are not parameters in a target zone. Expected inflation, $\mu_P$, cannot be constant here, as it would be under free floating, and neither can be $\sigma_P$.

The boundaries are *reflecting barriers* [3], in other words the monetary authority does not peg the exchange rate when it hits the boundaries, it simply intervenes not to let it go above $\overline{S}$, or below $\underline{S}$, Interventions are assumed to take place *only* at the boundaries. The possibility of *intra marginal interventions*, those undertaken inside the band, is excluded [4].

Our economy is populated by a large number of identical immortal Ricardian agents who maximize the expected discounted present value of lifetime utility of consumption and real money balances. It is assumed that the utility function of the representative agent belongs in the Hyperbolic Absolute Risk Aversion (HARA) class. As well known the isoelastic or Constant Relative Risk Aversion (CRRA), exponential, and quadratic utility are members of this class.

For the HARA class Merton [1971] has shown that there is a general solution for the dynamic consumption-portfolio problem. The general form of the HARA class is given in Merton [1971] as:

$$U(C) = \frac{1-\gamma}{\gamma}\left(\frac{\beta C}{1-\gamma} + \eta\right)^{\gamma}$$

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2. The process $\frac{dx}{x} = \mu\,dt + \sigma\,dz$ is called a Geometric Brownian Motion. The naturel logarithm of $x$, $y = \ln x$ follows the process: $dy = \mu_y\,dt + \sigma_y\,dz$, with $\mu_y = \mu - \frac{\sigma^2}{2}$ and $\sigma_y = \sigma$.

3. In a Reflecting barrier the process spends zero time, and returns in the interior of the interval as the light rays reflected on a mirror. See Karlin S. and Taylor H. [1981].

4. Intramarginal interventions are lately becoming very common in the EMS. Technically this possibility would make $\mu$ in our money supply process a function of M.

In our case:

$$C = c^{\alpha} m^{1-\alpha} \qquad \alpha > 0 \text{ and constant.}$$

Then the Cobb-Douglas and logarithmic forms can be obtained for particular values of the parameters. For example the log case can be obtained for $\gamma = 0$.

U(.) is an instantaneous utility function. Log is the natural logarithm, $c$ and $m$ are instantaneous consumption and real balances respectively.

## • Assets Markets

There are four securities in the instantaneous assets market:

i) An internationally traded bond with certain real rate of return $\rho$, and price $b$. The evolution of the rate of return is given by:

$$\frac{db}{b} = \rho\, dt, \qquad \rho > 0 \text{ and constant.}$$

The fact that $\rho$ is constant is implied by the assumptions of zero inflation abroad and purchasing power parity.

ii) A domestic nominal bond in zero net supply with certain nominal rate of return $i$, and price N that grows according to:

$$\frac{dN}{N} = i\, dt, \qquad i > 0.$$

$i$ will be determined endogenously by the condition $\omega_N = 0$. Both bonds are pure discount bonds (zero coupon bonds).

iii) Real money balances: $m = \dfrac{M}{P}$.

iv) Monetary transfers from the government, equal to the seignorage created from issuing money and unrelated to individual money holdings. The monetary transfers is the way money is injected into the economy and are therefore the driving force of price level changes. $m_T$ is the present value of future real monetary transfers the evolution of which is assumed to be governed by the diffusion:

$$\frac{dm_T}{m_T} = \mu_T\, dt + \sigma_T\, dz.$$

Later on we endogenously solve for the infinitesimal parameters $\mu_T$ and $\sigma_T$.

Real wealth W is the sum of *Monetary* wealth $M^t$ ($= m + m_T$) and portfolio wealth $f$ ($= r + B$). B is the total amount of real bonds owned by individuals in the economy.

In this Ricardian world the central bank's foreign bonds $r$ are viewed by the individuals as their own. Consequently since nominal bonds are in zero net supply:

$$W = f + m + m_T \qquad \text{and} \qquad \omega_f + \omega_m + \omega_N + \omega_T = 1.$$

where $\omega_f$, $\omega_m$, $\omega_N$, and $\omega_T$ are the shares of the individual's wealth held in the form of foreign bonds, money, domestic bonds, and monetary transfers respectively.

# 3 Optimal Consumption and Portfolio Choices Under An Exchange Rate Target Zone

The representative agent's problem is the following:

$$(3) \qquad J(W_0, S, t_0) = \operatorname{Max} E\left[\int_0^\infty e^{-\rho t} U(c, m)\, dt\right], \qquad \rho > 0$$

subject to:

$$(2) \qquad \frac{dS}{S} = \mu_P(M)\, dt + \sigma_P(M)\, dz, \qquad S_0 \text{ given}$$

and

$$(4) \quad dW = [(1 - \omega_m - \omega_N - \omega_T)\,\rho W + \omega_m(\sigma_P^2 - \mu_p)\,W + \omega_N(i - \mu_p + \sigma_p^2)\,W + \omega_T\,\mu_T\,W - c]\,dt + [\omega_T\,\sigma_T - (\omega_m + \omega_N)\,\sigma_P]\,W\,dz, \qquad W_0 \text{ is given.}$$

The constant subjective discount rate $\rho$ has been assumed to equal the real rate of return on the international bond.

S in the derived utility of wealth function J(...) refers to the entire path of the exchange rate. $W_0$ and $t_0$ denote initial wealth and initial time respectively.

We now show the derivation of (4). To find the dynamics of real wealth we first determine the real rate of return on the nominal bond $\frac{N}{P}$. A direct application Ito's lemma to $\frac{N}{P}$ gives:

$$\frac{d(N/P)}{(N/P)} = [i - \mu_p + \sigma_p^2]\,dt - \sigma_p\,dz.$$

Similarly the real return on money balances is:

$$\frac{d(1/P)}{(1/P)} = [-\mu_p + \sigma_p^2]\,dt - \sigma_p\,dz.$$

Notice that the return on money is equal to the return on a nominal bond that pays zero rate of interest. And the equation of motion of real wealth is immediately obtained as a weighted average of all assets' rates of return less consumption.

Writing (4) in compact notation we have:

$$dW = [\alpha_w W - c]\, dt + \sigma_W W\, dz.$$

The problem in the interior of the zone is *not the same as* in a pure exchange rate floating system. The nominal interest rate is not constant in a system like this. Thus S must be included as a state variable in the value function $J(W_0, S, t_0)$ of stochastic dynamic programming.

The Hamilton-Bellman-Jacobi (H-B-J) optimality equation for the above optimal stochastic control problem is:

$$0 = \max_{c\omega_m\, \omega_f\, \omega_N} [e^{-\rho t} U(c, m) + J_t + J_W(\alpha_W W - c) + J_S S \mu_p + \frac{1}{2} J_{WW} W^2 \sigma_W^2 + \frac{1}{2} J_{SS} S^2 \sigma_p^2 + J_{WS} \rho_{WS} \sigma_W W \sigma_p S]$$

We now define the undiscounted value function:

$$V(W_0, S) = e^{-\rho t} J(W_0, S, t_0).$$

taking advantage of the infinite horizon. The solutions of $V(..)$ are independent of time and are known as the stationary solutions.

The H-B-J equation can now be transformed to:

$$(5) \quad 0 = \max_{c\omega_m\, \omega_N\, \omega_T} [U(c, m) - \rho V(W_0, S) + V_W(\alpha_W W - c) + \frac{1}{2} V_{WW} W^2 \sigma_W^2 + V_S S \mu_p + \frac{1}{2} V_{SS} S^2 \sigma_p^2 + V_{WS} \rho_{WS} \sigma_W W \sigma_p S]$$

Differentiating (5) with respect to $c$, $\omega_m$, $\omega_N$, and $\omega_T$ we get the following first order conditions:

$$(F1) \quad U_c = V_W \quad \text{(envelope condition)}$$

$$(F2) \quad U_m = -V_W(-\rho - \mu_p + \sigma_p^2) - V_{WW} W (\sigma_p(\omega_N + \omega_m) - \omega_T \sigma_T)\sigma_p + V_{WS} \rho_{WS} S \sigma_p^2.$$

$$(F3) \quad V_W(i - \rho - \mu_p + \sigma_p^2) = -V_{WW} W (\sigma_p(\omega_N + \omega_m) - \omega_T \sigma_T)\sigma_p + V_{WS} \rho_{WS} S \sigma_p^2.$$

$$(F4) \quad V_W(-\rho + \mu_T) = V_{WW} W (\sigma_p(\omega_N + \omega_m) - \omega_T \sigma_T)\sigma_p - V_{WS} \rho_{WS} S \sigma_T.$$

Combining F1, F2, and F3 we obtain:

$$\frac{U_m}{U_c} = i$$

and

$$m^* = \frac{\alpha}{i} c^* \tag{6}$$

$$c^* = \frac{1}{V_W} \tag{7}$$

$$m^* = \frac{\alpha}{i V_W} \tag{8}$$

In this model as in STULZ R. [1984], PENATI A. and PENNACCHI G. [1988], and CLAESSENS C. [1986], consumption can be thought of as the divident paid from the stock of foreign bonds (tradable resources) $f$ (=B+$r$). And so the present value of the consumption stream is equal to $f$ (*non monetary wealth*). Similarly $m\,i$ can be viewed as the value of monetary services, the present value of which, from equation (6) is:

$$\alpha f \quad \text{(monetary wealth)}.$$

Consequently total wealth is $W = f + \alpha f = (1+\alpha) f$.

From our expression for monetary wealth we can calculate the value of future monetary transfers as the difference between total monetary wealth and money demand:

$$m_T = \frac{\alpha}{1+\alpha} W - \frac{\alpha}{i V_W}$$

We will utilize the above relationship in order to derive the process for the monetary transfers.

## • The Log Utility Case

We will now restrict our choice of the utility function to the isoelastic marginal utility (CRRA) family which includes the Cobb-Douglas and the logarithmic cases.

Given that even numerical solutions are difficult for the Cobb-Douglas case, and because we would like to focus on monetary factors since this is a monetary model with full price flexibility and no real effects, we proceed to solve the model for the case of log utility which separates real from monetary choices.

Thus

$$U(c, m) = \log c + \alpha \log m, \qquad \alpha > 0 \text{ and constant.}$$

$c$ and $m$ are instantaneous consumption and real money balances respectively.

MERTON [1971] has shown that for the above Bernoulli logarithmic utility function, the indirect utility function takes the following form [5]:

$$\text{(10)} \qquad V(W, S) = A \log W + \Phi(i, S)$$

Solving for equilibrium consumption and money holdings we have:

$$c^* = \rho f$$

$$m^* = \frac{\alpha \rho f}{i(M)}$$

Our solution for the demand for money is non-linear in levels, but in log form it will be similar to the Krugman form.

This is a consequence of the log utility assumption. In general for the HARA class the money demand will not be as simple, but it will have to include the function $\Phi(s)$.

This is easy to see from MERTON [1971], where he shows that the value function for the general HARA class with two state variables takes the following form:

$$J(W, S) = \frac{1-\gamma}{\gamma}\left(\frac{W}{1-\gamma} + \frac{\eta}{\beta\rho}\right)^{\gamma} \Phi(s)$$

And as a result optimal consumption takes the form:

$$C^* = \frac{1-\gamma}{\beta}\left[\left(\frac{W}{1-\gamma} + \frac{\eta}{\beta\rho}\right)\left(\frac{\Phi(s)}{\beta}\right)^{\frac{1}{\gamma-1}} - \eta\right]$$

Since in our case $m^* = \frac{\alpha}{i} c^*$, $m^*$ will involve the function $\Phi(s)$ as well. Thus we are showing that our methodology has the potential to provide interesting specifications for the demand for money.

Interest rates are ultimately functions of M since S is a function of M. The derivation of equilibrium consumption and money holdings is shown in appendix 1, section 1.1.

We know that $f$ (the stock of foreign bonds) evolves according to:

$$df = [\rho f - c]\, dt,$$

$\frac{df}{dt}$ can be interpreted as the balance of payments which in this economy is equal to income minus consumption, and since consumption in equilibrium equals $\rho f$,

$$\frac{df}{f} = 0.$$

5. See MERTON R. [1971].

So that we always have balance of payments equilibrium.

The process for the monetary transfers is given by:

$$\frac{dm_T}{m_T} = \left[\frac{(\mu_p - \sigma_p^2)(1+\alpha)}{(i\,V_W\,W - (1+\alpha))}\right] dt + \left[\frac{(1+\alpha)\,\sigma_p}{(i\,V_W\,W - (1+\alpha))}\right] dz.$$

The above follows basically from the definition of monetary wealth and its properties. The derivation can be found in appendix 1, section 1.2. It is now easy to see that the diffusion coefficient of real wealth vanishes making wealth riskless in equilibrium (see appendix 1, section 1.3.).

From (F3) after substituting for $\omega_m$, $\omega_T$, $\sigma_T$, $\sigma_{WS}=0$ (since wealth is riskless), we have:

(11) $$i = \rho + \mu_p\,(M) - \sigma_p^2\,(M).$$

The above equation for the rate of interest can be viewed as an interest rate parity condition. $\rho$ is the foreign interest rate and $\mu\,(M) = E\,\frac{dP}{P} = E\,\frac{dS}{S}$ is the expected depreciation of the domestic currency. It differs from the standard risk neutral equation since it includes a time-varying risk premium term on the domestic bond. Thus our interest rate equation is a more general one, since ours is a risk averse environment, that includes the Krugman risk neutral interest rate parity equation as a special case.

Since the exchange rate is endogenous, and assuming no explosive bubbles, its rational expectations solution is a function of the money supply: $S=S\,(M)$, and so interest rates are ultimately functions of the nominal money supply. Equation (11) is the stochastic version of the Fisher equation with variable expected inflation and diffusion coefficient $\sigma_p^2$. Under certainty $\sigma_p^2$ term vanishes. As a result in this model nominal interest rates are not constant, instead they become functions of the money supply and therefore *stochastic*. In a freely floating exchange rate regime or in a fixed exchange rate regime the function $\Phi\,(S)$ would be a constant one because S would not be a state variable. Consequently $i$ would be equal to $\rho+\mu-\sigma^2$ [6].

As a result of using (10) we have transformed the partial differential equation (5) into an ordinary one. For verification that (10) is the correct form for the value function see appendix 2. Notice that the terms in $V_{WW}$ and $V_{WS}$ in (5) disappear, the reason being that $\sigma_W=0$. In other words in equilibrium the uncertainty in real wealth disappears because the risk of holding money balances is exactly offset by the monetary transfers, the driving force behind price level movements.

We can solve for the function $S\,(M)$, the drift and diffusion of the price level process $\mu_p$ and $\sigma_p$ and nominal interest rates by equating real money

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6. Fisher S. [1975], Claessens C. [1986].

supply $\frac{M^S}{S}$ and real money demand $m$.

$$\frac{M}{S} = \frac{\alpha \rho f}{E\left(\frac{1}{S}\frac{dS}{dt}\right) + \rho - \sigma_S^2 (M)}.$$

Using Ito's lemma we find that exchange rates obey the following stochastic differential equation:

$$dS = \left[S'(M)\,\mu M + \frac{1}{2} S''(M)\,\sigma^2 M^2\right] dt + S'(M)\,\sigma M\, dz$$

Thus:

$$\mu_p (M) = \frac{S'(M)}{S(M)}\,\mu M + \frac{1}{2}\,\frac{S''(M)}{S(M)}\,\sigma^2 M^2.$$

and

$$\sigma_p (M) = \frac{S'(M)}{S(M)}\,\sigma M.$$

Substituting for the drift and the diffusion from above, into the money market equilibrium condition we get:

$$\frac{M}{S(M)} = \frac{\alpha \rho f}{\frac{S'(M)}{S(M)}\,\mu M + \frac{1}{2}\,\frac{S''(M)}{S(M)}\,\sigma^2 M^2 + \rho - \left(\frac{S'(M)}{S(M)}\,\sigma M\right)^2}$$

And the differential equation describing the dynamics of the exchange rate is:

$$\text{(12)}\quad \frac{1}{2}\,\frac{S''(M)}{S(M)}\,\sigma^2 M^3 + \frac{S'(M)}{S(M)}\,\mu M^2 + \rho M - \left(\frac{S'(M)}{S(M)}\right)^2 \sigma^2 M^3 - \alpha \rho f\, S(M) = 0$$

Subject to the following boundary conditions:

(12 $a$) $\quad S'(\underline{M}) = 0, \quad S'(\overline{M}) = 0$ (Smooth pasting [7])

(12 $b$) $\quad \overline{S} = S(\overline{M}), \quad \underline{S} = S(\underline{M})$ (Value matching)

The above equation (12) as we saw was derived from the money market equilibrium condition $\frac{M}{P} = m$ (real money supply equals real money demand). Equation (12) turns out to be non-linear due to the non linearity of the money demand function. Unlike this equation the Krugman differential

7. For discussions of the smooth pasting conditions see: SAMUELSON P. [1970], KRUGMAN P. [1988], DUMAS B. [1988], FROOT K. and OBSTFELD M. [1988].

equation for the exchange rate is derived from a general linear asset pricing equation of the form:

$$S(k) = K + a\,E\,\frac{ds(k)}{dt}$$ and an application of Ito's lemma.

The equation is highly nonlinear and an analytical solution was not found, but it was solved numerically [8]. The function S (M) giving the *rational expectations* equilibrium exchange rate path, the drift $\mu_p$ (M), the diffusion coefficient $\sigma_p$ (M) (risk premium), and the nominal rate of interest $i$ (M) appear in figures one, two, three, and four respectively [9]. Nominal interest rates and the drift of the price level diffusion are decreasing nonlinear functions of the money supply M.

# 4 Conclusions

We have provided a model based on microeconomic principles that can account for all the main results of the monetary models of target zones.

An interesting extension in our view would be to solve the model for other members of the isoelastic utility class, in particular the Cobb-Douglas case which would allow for real effects in consumption and would produce a more interesting specification for the demand for money.

Another issue of interest which we have not addressed here, is the collapse of the regime. It is well known that the solution of the regime collapse problem under uncertainty, unlike the deterministic problem, involves the Krugman paradox.

We believe that this paradox can be naturally resolved within our framework. The amount of foreign reserve borrowing that the central bank which issues the "strong" currency needs in order to prevent a collapse in the "wrong" direction, can be endogenously derived from our optimization framework. We leave these issues though for future research.

8. The equation was solved using the IMSL routine DB2PFD which is a variable order, variable step size finite difference method with differed corrections. The following set of parameters was assumed: $\rho = 0.15$, $\mu = 0.30$, $\sigma = 0.50$, $\underline{M} = 500$, $\overline{M} = 600$, $\alpha = 0.30$, $f = E + 18$.

9. Strictly speaking given the following process:

$\frac{dP}{P} = \mu_p(S)\,dt + \sigma_p(S)\,dz$, the drift is defined as $P\,\mu_p(S)$ and the diffusion as $P^2\,\sigma_p^2(S)$.

## 1.1. Derivation of Equilibrium Consumption and Money Demand

Given (10), $\mathrm{A} = \dfrac{1+\alpha}{\rho}$. We know that $\mathrm{W} = (1+\alpha)f$ and $\mathrm{V_W} = \mathrm{A}\dfrac{1}{\mathrm{W}}$.
From the above:

$$c^* = \rho f$$

$$m^* = \frac{\alpha \rho f}{i}.$$

## 1.2. Derivation of Monetary Transfers Process

*Monetary* Wealth $\mathrm{M}^t = m + m_\mathrm{T}$. Taking natural logarithms and then the differential of ln $\mathrm{M}^t$ we have:

$$\frac{d\mathrm{M}^t}{\mathrm{M}^t} = \frac{1}{\mathrm{M}^t}\, dm_\mathrm{T} + \frac{1}{\mathrm{M}^t}\, dm.$$

and

$$\frac{d\mathrm{M}^t}{\mathrm{M}^t} = \frac{dm_\mathrm{T}}{m_\mathrm{T}}\,\frac{m_\mathrm{T}}{\mathrm{M}^t} + \frac{dm}{m}\,\frac{m}{\mathrm{M}^t}.$$

And since $\mathrm{M}^t$ is a constant fraction of $f$, it grows at the same rate as $f$ that is:

$$\frac{d\mathrm{M}^t}{\mathrm{M}^t} = \frac{df}{f} = 0. \qquad (df = [\rho f - c^*]\, dt).$$

Therefore:

(9)
$$\frac{dm_\mathrm{T}}{m_\mathrm{T}} = -\frac{dm}{m}\,\frac{m}{m_\mathrm{T}}.$$

$$m_\mathrm{T} = \frac{\alpha}{1+\alpha}\mathrm{W} - \frac{\alpha}{i\,\mathrm{V_W}}.$$

From (9) we get:

$$\frac{dm_\mathrm{T}}{m_\mathrm{T}} = \left[\frac{(\mu_p - \sigma_p^2)(1+\alpha)}{i\,\mathrm{V_W}\,\mathrm{W} - (1+\alpha)}\right] dt + \left[\frac{\sigma_p(1+\alpha)}{i\,\mathrm{V_W}\,\mathrm{W} - (1+\alpha)}\right] dz.$$

## 1.3. Proof that in Equilibrium Wealth is Riskless

We can also show that the diffusion coefficient of real wealth $\mathrm{W}\sigma_\mathrm{W}$ vanishes because $\sigma_\mathrm{W} = 0$.

$$\sigma_\mathrm{W} = (\omega_\mathrm{T}\,\sigma_\mathrm{T} - (\omega_\mathrm{N} + \omega_m)\,\sigma_p)$$

$$= \left[\frac{\alpha(i\,\mathrm{V_W}\,\mathrm{W} - (1+\alpha))}{i\,\mathrm{V_W}\,\mathrm{W}(1+\alpha)}\right]\left[\frac{\sigma_p(1+\alpha)}{i\,\mathrm{V_W}\,\mathrm{W} - (1+\alpha)}\right] - \left[\frac{\alpha\sigma_p}{i\,\mathrm{V_W}\,\mathrm{W}}\right] = 0.$$

## Verification of The Value Function Form

Differentiating the function V (.,.) we have:

$$V_{WW} = -\frac{A}{W^2}, \qquad V_S = \Phi'(S), \qquad V_{SS} = \Phi''(S), \qquad V_{WS} = 0.$$

From (10) and substituting the values for $c$, $m$, $\omega_m$, $\omega_T$, $\mu_T$, $\sigma_T$ and setting, $\omega_N = 0$ – since the nominal bonds are zero net supply – into equation [4] we verify that the form of the value function is correct (wealth drops out):

$$\text{(13)} \qquad \frac{1}{2}\Phi''(S)\,S^2\sigma_p^2 + S\,\mu_p\,\Phi'(S) - \rho\,\Phi(S) + A(i) = 0.$$

where

$$A(i) = (1+\alpha)\log\rho + (1+\alpha)(1-\alpha)$$

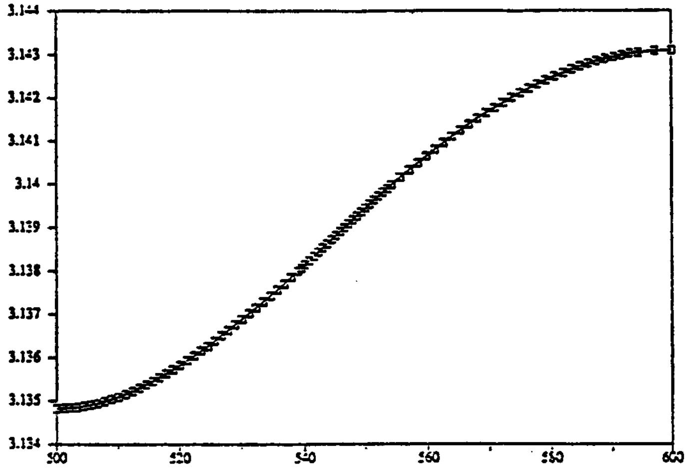
$$-1 - (1+\alpha)\log(1+\alpha) + \alpha\log\alpha - \alpha\log i(S) + 2\frac{\alpha\rho}{i(S)}$$

subject to:

$$\text{(13}a\text{)} \qquad \Phi'(\underline{S}) = \Phi'(\overline{S}) = 0.$$

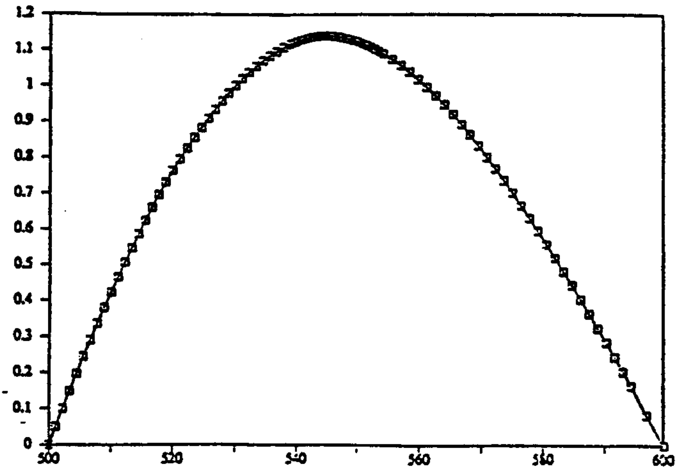
The above boundary conditions are again the smooth pasting or high order contact conditions.

FIGURE 1
***Nominal Exchange Rate***




***Nominal Money Supply***

FIGURE 2
***The Diffusion Coefficient***




***Nominal Money Supply***

FIGURE 3

***Expected Inflation***

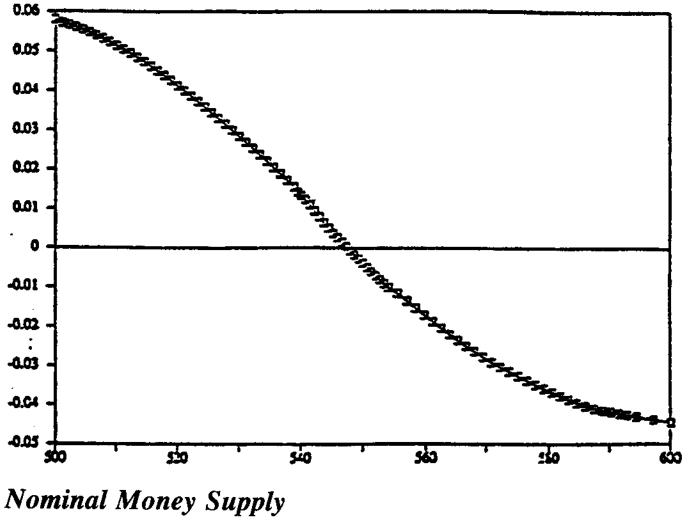
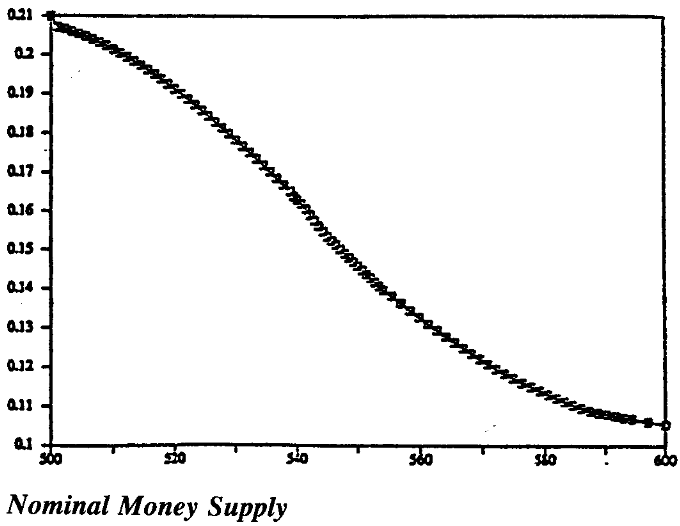



FIGURE 4

***Nominal Interest Rate***




# • References

ARTIS, M., TAYLOR, M. (1988). – "Exchange Rates, Interest Rates, Capital Controls and The European Monetary System: Assessing the Track Record" in GIAVAZZI, F., MICOSSI, S., MILLER, M., *The European Monetary System*, Cambridge University Press.

ATKINSON KENDALL, E. (1989). – "An Introduction To Numerical Analysis", John Wiley and Sons.

BELESSAKOS, E., LOUFIR, A. (1990). – "Has the European Monetary System Increased the Volatility of Nominal Interest Rates ?", *Mimeo*, University of Pennsylvania, July.

Claessens, C. (1986). – "Three Essays on Balance of Payments Crises", *Doctoral Dissertation*, University of Pennsylvania.

Delgado, F., Dumas, B. (1990). – "Monetary Contracting Between Central banks and the design of Sustainable Exchange Rate Zones", *Mimeo Wharton School*, University of Pennsylvania, March.

Dixit, A. (1988). – "A Simplified Exposition of Some Results Concerning Regulated Brownian Motion", *Mimeo*, Princeton University.

Dumas, B. (1988). – "Super Contract and Related Optimalty Conditions: A Supplement To Avinash Dixit's: A Simplified Exposition of Some Results Concerning Regulated Brownian Motion", *Mimeo Wharton School*, University of Pennsylvania, Nov.

Feenstra, R. (1986). – "Functional Equivalence Between Liquidity Costs And The Utility of Money", *Journal of Monetary Economics*, 17, pp. 271-291.

Fisher, S. (1975). – "The Demand For Index Bonds", *Journal of Political Economy*, 83, pp. 509-34.

Flood, P., Garber, P. (1983). – "A Model of Stochastic Process Switching", *Econometrica*, 51, N° 3, May.

Frenkel, J., Goldstein, M. (1986). – "A Guide To Target Zones", *IMF Staff Papers*, 33, N° 4, Dec.

Froot, K., Obstfeld, M. (1988). – "Nonlinear Indeterminacy in Linear Rational Expectations Models", *NBER Working paper*, Dec.

Froot, K., Obstfeld, M. (1989). – "Exchange Rate Dynamics Under Stochastic Regime Shifts: A Unified Approach", *NBER*, June.

Froot, K., Obstfeld, M. (1989). – "Stochastic Process Switching: Some Simple Solutions", *NBER Working paper*, June.

Giavazzi, F., Micossi, S., Miller, M. (1988). – "The European Monetary System", *Cambridge University Press.*

Harrison, J. M. (1985). – "Brownian Motion and Stochastic Flow Systems", New York: John Wiley and Sons.

Ingersol, J. (1987). – "Theory of Financial Decision Making", *Rowman And Littlefield.*

Karlin, S., Taylor, H. (1975). – "A First Course in Stochastic Processes", *Academic Press.*

Karlin, S., Taylor, H. (1981). – "A Second Course in Stochastic Processes", *Academic Press.*

Klein, M. (1989). – "Playing with the Band: Dynamic Effects of Target Zones in An Open Economy", *Manuscript Clark University*, Feb.

Krugman, P. (1988). – "Target Zones and Exchange Rate Dynamics", *NBER Working paper N° 2481*, Jan.

Krugman, P. (1989). – "Target Zones with Limited Reserves", *NBER Working paper*, Aug.

Merton, R. (1971). – "Optimum Consumption and Portfolio Rules in a Continuous Time Model", *Journal of Economic Theory* 3, pp. 373-413.

Miller, M., Weller, P. (1988). – "Target Zones Currency Options and Monetary Policy", *Mimeo*, University of Warwick, July.

Miller, M., Weller, P. (1988). – "Exchange Rate Bands and Realignments in a Stationary Stochastic Setting", *Mimeo*, University of Warwick, Sep.

Penati, A., Pennacchi, G. (1988). – "Optimal Portfolio Choice and the Collapse of a Fixed Exchange Rate Regime", *Mimeo*, The Wharton School, University of Pennsylvania, June.

Samuelson, P. (1970). – "The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variances and Higher Moments", *Review of Economic Studies*, March.

Sidrauski, M. (1967). – "Rational Choice and Patterns of Growth in a Monetary Economy", *American Economic Review*, 37, pp. 534-544.

Stulz, R. (1984). – "Currency Preferences, Purchasing Power Risks and the Determination of Exchange Rates in an Optimizing Model", *Journal of Money Credit and Banking*, 15, N° 3, Aug.

Svensson, L. E. (1989). – "Target Zones and interest Rate Variability: Where does the Variability Go, and Is a Fixed Exchange Rate Regime the Limit of a Narrow Target Zone", *Mimeo, Institute For International Economic Studies*, University of Stockholm, Aug.

Williamson, J., Miller, M. (1987). – "Target Zones and Indicators: A Blueprint for The International Coordination of Economic policy", *Institute For International Economics*, Sep.