ANNALES D'ÉCONOMIE ET DE STATISTIQUE. – N° 25/26 — 1992

Unique Equilibrium in a Model of Bargaining over Many Issues

Clara PONSATI-OBIOLS *

ABSTRACT. – A non-cooperative game of two parties bargaining an agreement involving *n* issues, each of which can be resolved in only two possible ways, is presented. Conditions under which there is a unique Perfect Bayesian Equilibrium (PBE) are provided. Along this equilibrium bargainers usually reach an agreement by means of trading a favourable solution on some of the issues in exchange for an unfavourable solution on the others, and some delay is necessary.

Unicité de l'équilibre dans un modèle de négociation comportant une multiplicité d'enjeux

RÉSUMÉ. – On discute un jeu non-coopératif modélisant une négociation entre deux personnes confrontées à *n* enjeux binaires. On propose des conditions sur lesquelles il existe un unique équilibre bayésien parfait. Souvent, les négociations réalisent un compromis à l'équilibre en échangeant une solution favorable sur un des enjeux contre une solution défavorable relativement à un autre enjeu, ce qui requiert un certain délai.

---

* C. PONSATI-OBIOLS : Universitat Autonoma de Barcelona, Departament d'Economia i Història Econòmica, Bellaterra (Barcelona) 08193, Espagne. Financial support from DGICYT grant PB-0294 is aknowledged.

1 Introduction

This paper studies how two parties bargain an agreement involving several issues when there is only two ways each issue can be resolved. Bargaining is modeled as a non-cooperative game taking place in continuous time: at each t, players know what issues remain unresolved, and they choose whether to remain firm or to yield on some of the issues, the bargaining goes on until all the issues have been resolved. Conditions under which this game has a unique Perfect Bayesian Equilibrium (PBE) are provided. Along this equilibrium bargainers usually reach an agreement by means of trading a favourable solution on some of the issues in exchange for an unfavourable (favourable to their opponent) solution on the others, and, if an agreement is reached, then it is necessarily after some delay.

Trading over different issues to reach a compromise is a widely extended mechanism of conflict resolution. In situations of bilateral trade over two indivisible commodities integer units of one good have to be traded for integer units of the other. In the context of labor-management bargaining, it is not uncommon to see a union accept a decrease in wage in exchange for an increase in job security or improved benefits. In the realm of political negotiations, civil wars are sometimes terminated when free elections and an amnesty are granted in exchange for the dismantlement of guerilla bases. In the regulatory arena, a government agency may impose higher safety standards to a public utility in exchange for an increase in the rate base or in the price caps. This are, however, phenomena that the bargaining literature has so far abstracted away.

The standard game-theoretic literature views bargaining as the problem of choosing a point in some joint utility possibility set: the particular way in which each issue is resolved does not matter, what matters is how much utility each party gets. If the joint utility possibility set is nonconvex, as it is when each issue has only a finite number of solutions, the standard approach requires the assumption that parties bargain not over utilities but over lotteries on the set of utilities. This has certainly been a fruitful approach. However, a more direct approach that deals explicitly with the resolution of each issue, may also be illuminating.

In the present model two parties bargain over several issues. Each issue can be resolved only in two possible ways, one favoured by each party. I assume that it is necessary to resolve all issues before an agreement can be implemented. And I assume that parties are impatient: given any agreement, both parties prefer it earlier rather than later. I assume moreover that the net value each party assigns to the resolution of each issue is private information and that a positive net value on each issue is necessary to obtain a utility that is preferred to disagreement. It is shown that if there is a positive probability that neither party is willing to yield on any issue (an agent will not be willing to yield on some issue if he gets a negative net value when the issue is resolved in his least preferred way) then there is a unique PBE. This result has been proved elsewhere (*see* PONSATI [1989]), for games with only one issue (*i.e.* games without any room for compromise). It will be shown here that the PBE of a two-issue game is uniquely characterized from the unique PBE of the one-issue games that arise as continuation games after one concession takes place. And it is shown that this construction of a unique PBE can inductively be carried on for any number of issues.

In addition to providing a strategic analysis explicitly addressing the problem of bargaining over several issues (*see* LINHART and RADNER [1988] for a non-strategic approach), the present is also, to my current knowledge, the first non-cooperative model of bargaining that yields a unique equilibrium in which any division of the surplus (any degree of compromise as well as total capitulation and permanent disagreement) can occur with positive probability, and agreements are reached only after some delay. The standard model, that of RUBINSTEIN [1982], has a unique Subgame Perfect Equilibrium in which a unique division of the surplus is reached without any delay. Since this seems to be at odds with the perception that delays and impasses are common in all kinds of negotiations, much effort has been made to show that, if there is private information, then the equilibrium outcomes may turn out to be inefficient ex-post because some types of players are willing to delay the agreement in order to signal their unwillingness to accept an unfavourable outcome. (*See*, for example, FUDENBERG, LEVINE and TIROLE [1985], FUDENBERG and TIROLE [1983], GROSSMAN and PERRY [1986], PERRY [1986], RUBINSTEIN [1985], and SOBEL and TAKAHASHI [1983]). However as GUL, SONNENSCHEIN and WILSON [1986] and GUL and SONNENSCHEIN [1988] point out, models of sequential bargaining with only one-sided private information cannot explain significant delays if proposals and counterproposals can be made quickly: in the limit, the informed party is able to obtain all the surplus without delay. Equilibria with both compromises and delays are possible in models with two-sided incomplete information (*see* AUSUBEL and DENEKERE [1988 *a*, *b*], CHO [1988], CRAMTON [1984, 1988], and PONSATI [1989]. However, the non-uniqueness of the equilibrium that prevents these models from having much predictive power appears to be an insurmountable obstacle.

Although at the expense of quite strong assumptions, the results seem to make the present exercise worthwhile. On the one hand we provide a model that has a very definite (albeit probabilistic) prediction: a unique pair of equilibrium strategies. On the other hand we confirm the intuitions that negotiations take time, and that, although compromises are a likely outcome, capitulations of one side and failures to reach an agreement are also possible. Thus, the old conventional wisdom that "anything is possible in bargaining" is reconciled with the aim to show that more specific predictions are possible.

The remainder of the paper is organized as follows: the general set up of the model is presented in Section 2, Section 3 presents the results for the one-issue bargaining game and the main result, relevant to games with n issues, is presented in Section 4 and some conclusions are presented in Section 5.

2 The Model

Consider two players ($i=1, 2$) bargaining on how to settle n different issues upon which they disagree. An agreement is denoted by a vector $\mathbf{x}=(\mathbf{x}_1, \mathbf{x}_2)=((x^1, \ldots, x^k, \ldots, x^n), (M^1-x^1, \ldots, M^k-x^k, \ldots, M^n-x^n))$ where each x^k can take one of two possible values, either x^{kH} or x^{kL} such that $0 \leqslant x^{kL} < x^{kH} \leqslant M^k$. An agreement can be reached at any time $t \in [0, \infty)$.

Let $(\mathbf{x}, t)$ denote an outcome of the bargaining, where $\mathbf{x}$ is an agreement and t the date at which the agreement is reached. Players preferences over pairs $(\mathbf{x}, t)$ are represented by utility functions $u^{\mathbf{s}_i}(\mathbf{x}, t)$ – where $\mathbf{s}_i=(s_i^1, \ldots, s_i^k, \ldots, s_i^n)$ are independent random variables whose realization is private information for each $i=1,2$. Player i's payoffs from outcome $(\mathbf{x}, t)$ when his type is $\mathbf{s}_i$ are as follows:

$$\text{(U.1)} \qquad u^{\mathbf{s}_i}(\mathbf{x}, t)=e^{-t}\prod_{\text{all } k} v_i^k(x_i^k-s_i^k) \qquad \text{if } x_i^k-s_i^k>0 \text{ for all } k,$$
$$<0 \qquad \text{otherwise,}^{1}$$

where v_i^k are differentiable, strictly increasing and concave.

Payoffs from disagreement are zero regardless of type. The ex-ante probability distribution of s_i denoted F_i is common knowledge and has densities f_i on $[0, x^{1H}] \times \ldots \times [0, x^{nH}]$. Let f_i^k denote the marginal density on s_{ik} and let f_i^{k-} denote the marginal density on

$$s_i^{k-}=(s_{i1}, \ldots, s_{ik-1}, s_{ik+1}, \ldots, s_{in}).$$

In Sections 3 and 4 we let $M^k=1$, $x^{kL}=(1-x^{kH})>0$, and $v_i^k(0)=0$ for convenience.

At each point in time each player chooses whether to yield on some of the issues or not. Once a player has yielded on an issue he cannot retreat. The game goes on until all issues have been resolved, that is, a concession has taken place on each issue. We assume that if a simultaneous concession takes place on some issue its resolution is selected randomly, being x^{kL} with probability π and x^{kH} with probability $(1-\pi)$. (The particular value of π will turn out to be irrelevant for our results, provided that $0<\pi<1$. We set $\pi=1/2$ for simplicity).

A *history of the game*, denoted h, is a list on how and when the issues have been resolved (with the convention that $h=\varnothing$ if none of the issues gets resolved). For example, let $n=5$, we could have $h=\{(x^{1L}, 3), (x^{3H}, 1), (x^{5L}, 3.5)\}$, indicating that no solution has been

1. The assumption that $u^{\mathbf{s}_i}(\mathbf{x}, t)<0$ if $x_i^k-s_i^k<0$ for some issue, where 0 is the utility generated by the perpetual disagreement outcome, is crucial for the results. On the other side I am working on the conjecture that the results can hold for a more general class of utility functions, provided that surpluses in different issues are not prefect substitutes.

agreed upon issues 2 and 4 (thus, disagreement prevails as the outcome of the bargaining), that player 2 conceded on issue 3 at $t=3$, and player 2 conceded on issues 1 and 5 at $t=3$ and $t=3.5$, respectively. A *history of the game at date t*, denoted h_t, is a date t, and a list of how and when issues have been resolved in the interval $[0, t)$, denoted $R(h_t)$. Given a history at date t, h_t, sometimes it will be useful to write it as $h_t=(h_t^j, h_t^i)$, to make explicit which elements of $R(h_t)$ are due to j's concession or to i's, respectively. Also, we will write $I(h_t)$ to indicate the set of issues for which no agreement has been reached. We will write h_0 to denote the history at $t=0$. Let H_t denote the set of all possible histories at t, and H denote the set of all possible histories.

A Δ-*reaction strategy for player i* is a measurable function

$$\varepsilon^i : S_i \times \bigcup_{t<\infty} H_t \rightarrow [0, \infty]^n,$$

$$\varepsilon^i(s_i, h_t) = [\tau^{i1}(s_i, h_t), \ldots, \tau^{in}(s_i, h_t)],$$

satisfying

(S.1) $\tau^{ik}(s_i, h_t) \in [t, \infty]$ for all k in $I(h_t)$.

(S.2) For each pair of histories h_t and $h_{t'}$ such that $R(h_t)=R(h_{t'})$ and $t<t' \leqslant \tau^{ik}(s_i, h_{t'})$ for all k, $\varepsilon^i(s_i, h_t)=\varepsilon^i(s_i, h_{t'})$.

(S.3) Let $h_{t+\delta}$ such that $R(h_t) \neq R(h_{t+\delta})$ for all $\delta>0$, then for all k in $I(h_{t+\delta})$, $\tau^{ik}(s_i, h_{t+\delta}) \in [t+\Delta, \infty)$ for all $\delta>0$.

Where $\varepsilon^i(s_i, h_t)=[\tau^{i1}(s_i, h_t), \ldots, \tau^{in}(s_i, h_t)]$ indicates that player i of type s_i, upon observing history h_t, concedes next on issue(s) k at date $\tau^{ik}(s_i, h_t)$. Condition (S.2) requires that players do not change their plans as long as they do not observe any new concession. Condition (S.3) states that it takes some time to react to new concessions: since $R(h_t) \neq R(h_{t+\delta})$ for all $\delta>0$ only if some player has made some concession(s) at t, then $\tau^{ik}(s_i, h_{t+\delta}) \in [t+\Delta, \infty)$ for all $\delta>0$ indicates that no new concession can take place in $(t, t+\Delta)$. We will think of the reaction lag Δ as negligible, and characterize the results in the limit as $\Delta \rightarrow 0$. However, it is necessary that $\Delta>0$ so that the "first" date at which players can react to events that took place at t is well defined.

Given a pair of types s and a history at t, h_t, each strategy profile ε generates a unique probability distribution over histories at t' for each $t'>t$, and hence a unique probability distribution over outcomes of the game (actually a unique history $h_{t'}$ and a unique outcome of the game if no simultaneous concessions are prescribed). By abuse of notation we will write $u^{s_i}(\varepsilon(s \mid h_t))$ to denote the (expected) utility that player i of type s_i obtains given the probability distribution over outcomes generated by $\varepsilon(s \mid h_t)$.

A *system of beliefs* for player i is a measurable function which maps each partial history into some probability measure on the types of the opponent,

$$B^i : \bigcup_{t<\infty} H_t \rightarrow \mathscr{P}\Big(\prod_{\text{all } k} [0, x^{kH}]\Big),$$

such that $B^i(h_0)_k(A)=\int_A f_j^k(z)\,dz$, where $B^i(h_t)_k$ denotes the projection of $B^i(h_t)$ on $[0, x^{kH}]$.

Consider the set of j's types

$$\varepsilon_j^{-1}(h_t)=\{s_j \text{ such that } \varepsilon_j(s_j, h_0) \text{ generates } h_t^j \text{ given } h_t^i\},$$

and let $\varepsilon_j^{-1}(h_t)_k$ be its projection on $[0, x^{kH}]$. A *system of beliefs* B *is consistent with a strategy profile* ε if it is the result of Bayesian updating. That is,

$$B^i(h_t)_k(A)=\frac{\int_{A\cap\varepsilon_j^{-1}(h_t)_k} f_j^k(z)\,dz}{\int_{\varepsilon_j^{-1}(h_t)_k} f_j^k(z)\,dz}$$

whenever the denominator is non zero.

Let $V^{s_i}(\varepsilon, h_t)$ denote the payoff that player i of type s_i expects from strategy profile ε, conditional on the fact the game has evolved up to h_t:

$$V^{s_i}(\varepsilon, h_t)=\int u^{s_i}(\varepsilon(s_i, \,.\, | h_t)\,dB^i(h_t).$$

A Perfect Bayesian Equilibrium (PBE) is a strategy profile ε^* and a system of beliefs B^* such that:

(E.1) For all h_t, for almost all types,

$$V^{s_i}(\varepsilon^*, h_t)\geq V^{s_i}(\varepsilon', h_t) \text{ for all } \varepsilon' \text{ such that } \varepsilon'^j=\varepsilon^{*j}, i,j=1,2.$$

(E.2) B^* is consistent with ε^*.

Conditions (E.1) and (E.2) require that strategies be best response to each other at each partial history, and that beliefs are updated according to Bayes's Rule whenever it applies.

3 The War of Attrition Game

Consider the situation in which only one issue is at discussion. The reader will readily recognize the game as a war of attrition, also known in the literature as a stopping game. The first player to concede and accept his least preferred solution will actually terminate the game. In this simpler game all that each player needs to do is to choose a date such that if no concession by the opponent has been observed, he will concede. There is an extensive literature on these kind of games (*see* KREPS and WILSON [1982], HENDRICKS, WEISS and WILSON [1988]). Here, we will only provide a discussion of the main results about wars of attrition that are relevant for our model (the reader is referred to PONSATI [1989] for a detailed discussion).

Consider a war of attrition with distributions of types F_i having a positive density f_i over the interval $[\underline{s}_i, x^H]$, with $0 \leq \underline{s}_i < x^L$. Consider an equilibrium strategy profile τ.

Let

$$G^i(t) = \text{Prob}(\tau_i^{-1}([0, t)) = \int_{\tau_i^{-1}([0,t))} f_i(z)\, dz,$$

and

$$\Gamma^i(t) = G^i(t) - \lim_{\delta \to 0} G^i(t-\delta),$$

$G^i(t)$ is the probability that i concedes no latter than t and $\Gamma^i(t)$ is the probability that i concedes at t.

A strategy profile τ is a Nash Equilibrium iff for $i=1, 2$, and f^i-a.e.,

$$\tau^i(z) \in \underset{t \in R^+}{\text{argmax}} \int_{[0,t)} (x^H - z) e^{-\omega} dG^j(\omega) + \Gamma^j(t) \left(\frac{x^H + x^L}{2} - z \right) e^{-t} + (1 - G^j(t))(x^L - z) e^{-t}.$$

Proposition 1 characterizes the set of NE for games with one issue.

PROPOSITION 1: For the game with only one issue there is a unique NE strategy τ. The strategy profile τ is characterized by a pair differentiable mappings

$$\sigma_i : (0, \infty) \to [\underline{s}_{il}, x^H], \qquad i = 1, 2$$

such that

$$\tau^i(s) = t \qquad \text{iff} \quad s_{ik} = \sigma_{ik}(t).$$

At $t=0$, at most one of the players concedes with positive probability, that is, if $\sigma_i(0) > \underline{s}_i$ for some i, then $\sigma_j(0) = \underline{s}_j$.

In order to proof Proposition 1 we will use the following.

LEMMA 2: Consider a PBE ε^*, then, for all $T < \infty$, G^i $i = 1, 2$ are differentiable *a.e.* in $(0, T)$.

Proof: See Appendix.

Lemma 2 has the following intuitive explanation. Assume that G^i failed to be absolutely continuous at some point t, $t \in (0, \infty)$. That is, G^i would be growing infinitely fast at that point. If this where the case, there would be an interval of time arount t such that player j would benefit from waiting and making a concession after this interval because the payoff that he would lose if he delayed his concession would be bounded while the payoff that he would gain would not (because the probability of i's concession is increasing without bound). However, why should G^i be increasing at t when j makes no concession in the interval $[t-\delta, t+\delta]$? In fact, j would be better off concentrating any concession in the interval $[t-\delta, t+\delta]$ to date $t-\delta$. But this would imply G^i being flat at t, a contradiction.

LEMMA 3: At most one player makes a concession at $t=0$ with positive probability.

Proof: See Appendix

Proof of Proposition 1: If $G^i(t)$ is differentiable, then

$$\tau_j(s) \in \operatorname{argmax} \int_{[0,t)} v_j(x^H - s)\, e^{-\varpi}\, dG^i(\varpi) + (1 - G^i(t))\, v_j(x^L - s)\, e^{-t},$$

and for any interior $\tau_j(s)$ the following first order condition is necessary

$$dG^i(\tau_j(s))\,[v_j(x^H - s) - v_j(x^L - s)] = (1 - G^i(\tau_j(s)))\, v_j(x^L - s).$$

Notice that in an equilibrium no $s > x^L$ concedes. Also, let us check that all types $s < x^L$ choose to concede at some finite date. Let $\alpha = \text{Prob}(s_j > x^L)$, if player i of type z does not concede he receives a payoff

$$\int_{[0,\infty)} (x^H - z)\, e^{-\tau}\, dG^j(\tau),$$

notice that for each $t > 0$,

$$\int_{[0,\infty)} (x^H - z)\, e^{-\tau}\, dG^j(\tau) \leqslant \int_{[0,t)} (x^H - z)\, e^{-\tau}\, dG^j(\tau) + (x^H - z)\, e^{-t}(\alpha - G^j(t)),$$

because the right hand side is what player i can receive if he does not concede and j concentrates at all the probability of conceding at t or later. Notice that for each $\gamma > 0$, for t large enough, $\alpha - G^j(t) < \gamma$. Therefore, if $z < x^L$, then there is some $t < \infty$ such that

$$\int_{[0,t)} (x^H - z)\, e^{-\tau}\, dG^j(\tau) + (x^H - z)\, e^{-t}(\alpha - G^j(t))$$
$$< \int_{[0,t)} (x^H - z)\, e^{-\tau}\, dG^j(\tau) + (x^L - z)\, e^{-t}(1 - G^j)(t)),$$

hence

$$\int_{[0,\infty)} (x^H - z)\, e^{-\tau}\, dG^j(\tau) < \int_{[0,t)} (x^H - z)\, e^{-\tau}\, dG^j(\tau) + (x^L - z)\, e^{-t}(1 - G^j(t)),$$

i.e. conceding at t is better than not conceding. Moreover it cannot be the case that for some i, all $\mathbf{s}_i$ that concede choose to do it at $t=0$: in this case any $\mathbf{s}_j$ observing no concession at $t=0$ should concede as soon as possible after 0, but then any $\mathbf{s}_i$ would be better off not conceding at 0.

Thus, in any equilibrium $\tau_j(s)$ takes values in the interior of $[0, \infty]$. For all $t \in (0, \infty)$ let

$$\tau_i(s_i)=t \qquad \text{iff} \quad s_i=\sigma_i(t).$$

Then $G^i(t)=F_i(\sigma_i(t))$, and $dG^i(t)=\sigma_i'(t)f_i(\sigma_i(t))$. Hence $\sigma_i(.)$, $i=1,2$ must be a solution to the system of differential equations

$$(1) \quad \sigma_i'(t)f_i(\sigma_i(t))[v_j(x^H-\sigma_j(t))-v_j(x^L-\sigma_j(t))] = [1-F_i(\sigma_i(t))]\,v_j(x^L-\sigma_j(t)),$$

$$i,j=1,2, \qquad i \neq j,$$

There are many such solutions. However only one will be compatible with the remaining necessary condition that any equilibrium must satisfy. For each s_1 consider the type s_2 such that $\tau_1(s_1)=\tau_2(s_2)$, this relationship defines a function on the rectangle $[\underline{s}_1, x^H] \times [\underline{s}_2, x^H]$. Let it be denoted by Ψ. Notice that

$$\Psi'(\sigma_1(t))=\frac{\sigma_1'(t)}{\sigma_2'(t)}$$

and that $\Psi(x^L)=x^L$. There is a unique such Ψ. Since

$$\lim_{t\to 0}\sigma_i(t)=\lim_{t\to 0}\sigma_j(t)$$

must lie on Ψ, we have a unique initial condition that the relevant solution to (1) must satisfy.

Does this unique solution to (1) actually give rise to an equilibrium strategy profile? Notice that if $s=\sigma_i(t)$ concedes at any $t'>t$

$$dG^i(t')[v_j(x^H-s)-v_j(x^L-s)]<(1-G^i(t'))\,v_j(x^L-s),$$

because

$$dG^i(t')[v_j(x^H-s')-v_j(x^L-s')]=(1-G^i(t'))\,v_j(x^L-s'),$$

for some $s'=\sigma_i(t')>s$ and, since v_j is concave, $\dfrac{v_j(x^L-z)}{v_j(x^H-z)-v_j(x^L-z)}$ is decreasing. That is, the payoff to $s=\sigma_i(t)$ if he concedes at any $t'>t$ is decreasing, while if he concedes at $t''<t$ the payoff is increasing. Thus, conceding at t is a payoff maximizing decision. □

4 Bargaining Over Many Issues

In this section we study the game with many issues. The crucial step in characterizing the set of PBE is to show that the date at which players chose to make the first concession does not reveal their type on the remaining issues, thus the information assumptions that we have made on the initial situation (a positive density on some set $[\underline{s}_{i1}, x^{1\mathrm{H}}] \times \ldots \times [\underline{s}_{in}, x^{n\mathrm{H}}]$, with $0 \leqslant \underline{s}_{ik} < x^{k\mathrm{L}}$) are kept in the $n-1$ issues continuation game. Thus from our characterization of the game with one issue we can inductively characterize the game with many issues. Our main result is the following.

Proposition 4: For any game with n issues such that beliefs at $t=0$ have a positive density on some set $[\underline{s}_{i1}, x^{1\mathrm{H}}] \times \ldots \times [\underline{s}_{in}, x^{n\mathrm{H}}]$, with $0 \leqslant \underline{s}_{ik} < x^{k\mathrm{L}}$, $i=1,2$, there is a unique PBE strategy profile. The unique PBE is fully characterized by a pair of differentiable mappings

$$\sigma_i : (0, \infty) \rightarrow [\underline{s}_{i1}, x^{1\mathrm{H}}] \times \ldots \times [\underline{s}_{in}, x^{n\mathrm{H}}], \qquad i=1,2$$

such that

$$\tau^{ik}(s) = t \text{ iff } s_{ik} = \sigma_{ik}(t) \text{ and, for all } r \neq k,\ s_{i_r} \geqslant \sigma_{i_r}(t).$$

At $t=0$, at most one of the players concedes with positive probability, that is, if $\sigma_{ik}(0) > \underline{s}_{i\mathrm{K}}$ for some i and k, then $\sigma_{jk}(0) = \underline{s}_{jk}$ for all k.

In Section 3 we have seen that the proposition is true for $n=1$. The induction argument will be in three steps. First we will show that for any PBE strategy profile ε, the $n-1$ issues games that start after any first concession satisfy the hypothesis of Proposition 4 because, with respect all other issues, a concession on issue k at t reveals nothing about the type of the player who conceded. Hence, by the induction hypothesis, the value of the continuation game is unique. Next we will see that the equilibrium must be characterized by first order conditions which in turn imply the existence of differentiable mappings

$$\sigma_i : (0, \infty) \rightarrow [\underline{s}_{i1}, x^{1\mathrm{H}}] \times \ldots \times [\underline{s}_{in}, x^{n\mathrm{H}}], \qquad i=1,2,$$

such that

$$\tau^{ik}(s) = t \qquad \text{iff} \quad \sigma_{ik}(t) = s_{ik} \quad \text{and for } r \neq k\, \sigma_{i_r}(t) \leqslant s_{i_r}.$$

Finally we will prove that for any initial beliefs with a positive density on $[\underline{s}_{i1}, x^{1\mathrm{H}}] \times \ldots \times [\underline{s}_{in}, x^{n\mathrm{H}}]$, with $0 \leqslant \underline{s}_{ik} < x^{k\mathrm{L}}$, there is a unique mapping σ *i.e.* $\varepsilon(h_0)$ is uniquely defined. Thus, we will have a unique PBE strategy profile.

Proof of Proposition 4: We first introduce some notation. Let h_t^n denote the history at t such that no concession has occurred in $[0, t)$. Given a strategy profile ε, consider

$$G^j(t) = \text{Prob}(\varepsilon_j^{-1}(h_t^n)) = \int_{\varepsilon_j^{-1}(h_t^n)} f_j(z)\, dz.$$

We will write $\varepsilon^i(\mathbf{s}_i, h_0)$ as $\varepsilon^i(\mathbf{s}_i) = [\tau^{i1}(\mathbf{s}_i), \tau^{i2}(\mathbf{s}_i), \ldots, \tau^{in}(\mathbf{s}_i)]$, that is the vector of dates at which player i of type $\mathbf{s}_i$ plans to concede provided that he has not observed concessions by j. We will write $V_i^{fk-}(\mathbf{s}_i \mid t)$ to denote the value to player i of type $\mathbf{s}_i$ in the continuation game that starts after j has conceded on issue k at date t, and $V_i^{yk-}(\mathbf{s}_i \mid t)$ to denote the value to player i of type $\mathbf{s}_i$ in the continuation game that starts after i has conceded on issue k at date t.

- **Step 1**

Here we will show that the optimal date of i's concession for issue k depends only on component s_{ik}. Thus, at $t=0$, players compute the optimal dates to concede on each issue and follow this calendar of concessions as long as no concession by the opponent is observed. After the first concession takes place by i's conceding k, the continuation $(n-1)$–issue game satisfies the hypothesis of Proposition 4 because the date at which player i concedes on issue k depends only on s_k.

Assume that player i makes his first concession on issue k. Then, given the strategy of his opponent, ε_j, he will choose a date of concession $t_{s_i}^k$ to maximize

$$\int_{[0,\,t)} \sum_{\text{all } k} \alpha_\tau^{jk} v_i^k (x^{kH} - s_{ik}) V_i^{fk-}(\mathbf{s}_i \mid \tau)\, dG^j$$
$$+ \Gamma^j(t) [1/2 \sum_{\text{all } k} \alpha_t^{jk} v_i^k (x^{kH} - s_{ik}) V_i^{fk-}(\mathbf{s}_i \mid t) + 1/2\, v_i^k (x^{kL} - s_{ik}) V_i^{yk-}(\mathbf{s}_i \mid t)]$$
$$+ (1 - G^j(t))\, v_i^k (x^{kL} - s_{ik}) V_i^{yk-}(\mathbf{s}_i \mid t),$$

where $\Gamma^j(t)$ denotes the size of the atom t and α_τ^{jk} is the probability that j concedes on issue k at τ, conditional on the event that he concedes at τ. For each strategy profile ε, and for each t, consider the set of types

$$I_i^k(t \mid \varepsilon_j) = \{\mathbf{si}_i \text{ such that } \tau^{ik}(\mathbf{s}_i) = t\},$$

that is, the set of types of i such that if no concession by j occurs, will make their concession on k at date t. Lemma 7, that we prove using Lemma 6, shows that whether or not $\mathbf{s}_i$ lies in $I_i^k(t \mid \varepsilon_j)$ depends only on s_{ik}.

LEMMA 5: No Unilateral Intransigence. In any PBE, if $s \ll x^L$, then $\tau^{ik}(s) < \infty$ for all k.

Proof: See Appendix.

Lemma 5 rules out strategies in which a player refuses to make any concession before the opponent yields first on some issue.

LEMMA 6: Weak Monotonicity. In any PBE, if $s \ll s$, then $\tau^{ik}(\mathbf{s}') \leqslant \tau^{ik}(\mathbf{s}')$.

Proof: See Appendix.

LEMMA 7: Independence of the Issues. For any PBE strategy profile ε, whether or not s_i lies in $I_i^k(t \mid \varepsilon_j)$ depends only on s_{ik}.

Proof: See Appendix.

Thus, with respect all unresolved issues, although a concession on issue k at t reveals exactly the player's type on issue k, it reveals nothing about the type of the player who made it. That is, the continuation game satisfies the hypothesis of Proposition 4 and therefore has a unique continuation PBE.

- **Step 2**

Next we see that, as in a simple war of attrition, G^j must be a differentiable function in $(0, \infty)$ and at most one of the players concedes at $t=0$ with positive probability. Moreover, the values of the continuation games, $V^{k-}(s \mid t)$, must also be differentiable. Thus a PBE strategy profile will be characterized by first order conditions. We state this as Lemmas 8 and 9 which are proved in the appendix. The intuitive arguments are the same as in section 3.

LEMMA 8: Consider a PBE ε, then, for all $T<\infty$, G^i and $V_i^{fk-}(s_i \mid .)\, i=1,2$ are differentiable *a.e.* in $(0, T)$.

Proof: See Appendix.

LEMMA 9: The set of types of a player that concede at $t=0$ has positive measure for at most one player.

Proof: See Appendix.

- **Step 3**

We can now complete the proof of Proposition 4.

For each $s_t \in [\underline{s}_{i1}, x^{1H}] \times \ldots \times [\underline{s}_{in}, x^{nH}]$, let

$$S_{ti} = [s_{ti1}, x^{1H}] \times \ldots \times [s_{tin}, x^{nH}], \qquad S_{ti}^k = [s_{tik}, x^{kH}],$$

and

$$S_{ti}^{k-} = [s_{ti1}, x^{1H}] \times \ldots \times [s_{tik-1}, x^{k-1H}] \times [s_{tik+1}, x^{k+1H}] \times \ldots \times [s_{tin}, x^{nH}].$$

By Lemmas 5 to 7, after a concession on issue k at t players beliefs on types s_i^{k-}, will be updated as the restriction of the original distribution to S_{ti}^{k-} for some s_t. That is, by the induction assumption, an equilibrium will be fully characterized by s_{ti}, $i=1,2$ as functions of time. Let σ_i denote such functions. Since

$$(1-G^j(t)) = \int_{S_{tj}} f_j(z)\, dz, \tag{2}$$

and since $G^j(t)$ is differentiable with respect to t *a.e.* for $t>0$, we will have that σ_i must be differentiable with respect to t *a.e.* for $t>0$. Then

$$dG^j(t) = -\sum_{\text{all } k} \sigma'_{jk}(t) f_j^k(\sigma_{jk}(t)) \int_{S_{tj}^{k-}} f_j^k(z)\, dz. \tag{3}$$

Since $\sigma_i(t)$ and $V_i^{yk-}(\mathbf{s}_i \mid t)$ are differentiable with respect ot t *a.e.* for $t>0$, for any interior $t_{\mathbf{s}_i}^k$ the following first order condition is necessary *a.e.*

$$(4)\quad dG^j(t_{\mathbf{s}_i}^k)\Big\{\sum_{\text{all } k} \alpha_{t_{si}^k}^{jk}\, v_i^k(x^{kH}-s_{ik})\,V_i^{fk-}(\mathbf{s}_i \mid t_{\mathbf{s}_i}^k) - v_i^k(x^{kL}-s_{ik})\,V_i^{yk-}(\mathbf{s}_i \mid t_{\mathbf{s}_i}^k)\Big\}$$
$$+(1-G^i(t_{\mathbf{s}_i}^k))\,v_i^k(x^{kL}-s_{ik})\frac{\partial V_i^{yk-}(s_i \mid t_{\mathbf{s}_i}^k)}{\partial t}=0.$$

In particular for $\mathbf{s}_{ti}=\sigma_i(t)$ we must have

$$(5)\quad dG^j(t)\Big\{\sum_{\text{all } k} \alpha_t^{jk}\, v_i^k(x^{kH}-s_{tik})\,V_i^{fk-}(\mathbf{s}_{ti} \mid t) - v_i^k(x^{kL}-s_{tik})\,V_i^{yk-}(\mathbf{s}_{ti} \mid t)\Big\}$$
$$+(1-G^i(t))\,v_i^k(x^{kL}-s_{tik})\frac{\partial V_i^{yk-}(\mathbf{s}_{ti} \mid t)}{\partial t}=0 \text{ for all } k.$$

Moreover

$$(6)\qquad \alpha_t^{jk}=\frac{\displaystyle\int_{\mathbf{S}_{tj}^k} f_j(z, s_t^{k-})\,dz}{\displaystyle\sum_{\text{all } k}\int_{\mathbf{S}_{tj}^k} f_j(z, s_t^{k-})\,dz}.$$

And recall that, by Lemma 9, for all k,

$$(7)\qquad \begin{cases} \lim_{t\to 0}\sigma_{ik}(t)\geqslant s_{ik}, \text{ with equality for at least one } i=1,2, \\ \qquad\qquad \text{and } \lim_{t\to\infty}\sigma_{ik}(t)=x^{kL}, \end{cases}$$

since no $\mathbf{s}_i$ with $s_{ik}>x^{kL}$ chooses a finite date of concession on issue k.

Substituting (2), (3), and (6) into (5) we get

$$(8)\quad \Big\{\sum_{\text{all } k}\sigma_{jk}'(t) f_j^k(\sigma_{jk}(t))\int_{\mathbf{S}_{tj}^{k-}} f_j^k(z)\,dz\Big\}$$
$$\times\left\{\sum_{\text{all } k}\frac{\displaystyle\int_{\mathbf{S}_{tj}^k} f_j(z, s_t^{k-})\,dz}{\displaystyle\sum_{\text{all } k}\int_{\mathbf{S}_{tj}^k} f_j(z, s_t^{k-})\,dz}\, v_i^k(x^{kH}-\sigma_{ik}(t))\,V_i^{fk-}(\sigma_i(t) \mid t)\right.$$
$$\left. - v_i^k(x^{kL}-\sigma_{ik}(t))\,V_i^{yk-}(\sigma_i(t) \mid t)\right\}$$
$$=\int_{\mathbf{S}_{tj}} f_j(z)\,dz\, v_i^k(x^{kL}-\sigma_{ik}(t))\frac{\partial V_i^{yk-}(\sigma_i(t) \mid t)}{\partial t},$$
$$k=1,\ldots,n \qquad \text{and} \qquad i,j=1,2.$$

This system of differential equations, with conditions (7), yields a unique solution. The same argument used in section 3 shows that such a solution characterizes a PBE. □

EXAMPLES

Example 1: Bilateral Trade of Two Indivisible Commodities.

Let $n=2$ and consider the following problem of bilateral trade. Assume each player i owns 3 units of the indivisible commodity i, and let the preferences of both players on the space of goods and dates be represented by discounted Cob-Douglas utility functions.

$$v^i(\mathbf{z}-\mathbf{s}_i,\, t)=(z_1-s_{i1})(z_2-s_{i2})e^{-t},$$

where the vector of parameters s_i is private information. Thus the possible outcomes if trades takes place are :

I. 1 gets 2 units of each good and 2 gets only 1 of each,

II. 1 gets 2 units of 1 and 1 unit of 2, and 2 gets 1 unit of 1 and 2 units of 2,

III. 1 gets 1 units of 1 and 2 units of 2, and 2 gets 2 units of 1 and 1 unit of 2,

IV. 2 gets 2 units of each good and 1 gets only 1 of each.

Since players can abstain from consumption and get 0 utility, the type of any player participating in the bargaining, must be such that $(2-s_{i1})(2-s_{i2})\geqslant 0$, *i.e.* the best trade outcome is not worse than the no trade outcome: thus the assumption that $[0,\, 2]^2$ is the support of $\mathbf{s}_i$.

Example 2: Computing the PBE of a Symmetric Game.

Consider the following case. Let $x^{kL}=x^{L}$ and $u^k(z)=z$, and let s_{ik} be uniformly distributed in $[0,\, x^{H}]$ for all k. We will characterize the equilibrium in the limit as $\Delta\to 0$. By symmetry, for each t we have $\mathbf{s}_t=(s_t,\, \ldots,\, s_t)$. It will suffice to construct the mapping $\sigma(.)$ that gives for each the value of s_t. In general a player of type $\mathbf{s}$ will chose to concede on issue k at date $k^k(\mathbf{s})=\sigma^{-1}(s_k)$, provided that issue k is not resolved before $t^k(\mathbf{s})$.

Notice that since, in the limit as $\Delta\to 0$, type $\mathbf{s}_t$ concedes on all issues at date t, we have that $V_i^{yk-}(\mathbf{s}_{ti}\,|\,t)=V^{fk-}(\mathbf{s}_t\,|\,t)=(x^{L}-s_t)^{(n-1)}e^{-t}$. Notice also that since types s_{ik} are uniformly distributed in $[0,\, x^{H}]$, the probability of a concession on issue k at τ, given that a concession takes place at t, is $\alpha_\tau^k=1/n$ for all τ and k. Thus the first order condition (5) for type $\mathbf{s}_t$,

$$dG^j(t)\Big(\sum_{\text{all } k}\alpha_t^{jk}(x^{H}-s_{ti})V_i^{fk}(\mathbf{s}_{ti}\,|\,t)-(x^{L}-s_{ti})V_i^{yk-}(\mathbf{s}_{ti}\,|\,t)\Big\}+(1-G^i(t))(x^{L}-s_{ti})\frac{\partial V_i^{yk-}(\mathbf{s}_{ti}\,|\,t)}{\partial t}=0,$$

can be written as

$$dG^j(t)\{(x^{H}-x^{L})(x^{L}-s_t)^{(n-1)}e^{-t}\}-(1-G^i(t))(x^{L}-s_{ti})(x^{L}-s_t)^{(n-1)}e^{-t}=0.$$

Since $G^j(t)=\dfrac{\sigma(t)}{x^{H}}$, and $dG^j(t)=\sigma'(t)\dfrac{\sigma(t)}{x^{H}}$, we have that $\sigma(.)$ must solve the differential equation

$$\sigma'(t)(x^{H}-x^{L})-(x^{H}-\sigma(t))(x^{L}-\sigma(t))=0,$$

with initial condition

$$\sigma(0)=0.$$

It is easily checked that our solution $\sigma(.)$ is characterized by

$$\log\frac{x^{L}[x^{H}-\sigma(t)]}{x^{H}[x^{L}-\sigma(t)]}=t$$

That is, each player $\mathbf{s}$ concedes on issue k at $t^k(\mathbf{s})$ such that

$$t^k(\mathbf{s})=\log\frac{x^{L}[x^{H}-s_k]}{x^{H}[x^{L}-s_k]}.$$

5 Conclusions

We have presented a model of bargaining in which two players can reach an agreement by means of trading concessions on the different issues involved. We have made assumptions on preferences implying that the resolution of the different issues are not substitutes for one an other but rather they complement each other in the sens that an acceptable level is necessary on each issue. We have also assumed that there is two-sided uncertainty on what the opponent considers an acceptable solution for each issue. Under our assumptions we have proved that there is a unique pair of Perfect Bayesian Equilibrium strategies. Along these strategies each bargainer decides when to yield on each issue depending only on the parameter that measures the importance of that same issue. Therefore, although the model looks like a very complex bargaining game, it is actually played as a set of independent and simultaneous "wars of attrition", one for each issue.

Strategies of the form "I will concede on issue A only when you yield on issue B" are never part of an equilibrium: the intuition behind this fact is that such statements are incredible threats because a player willing to respond with concession on A to a nice opponent that yields on B, should be even more willing to concede to a nasty opponent that refuses to yield on B. In equilibrium players exchange statements of the form "Since I have yield on issue A, I now expect that you will be nice and yield on issue B". However, since the opponent may or may not be nice, the player that starts yielding on issue A may continue to yield on issue B and then say " "Since I have yield on issues A and B, I now expect that you will be nice and yield on issue C", etc.

LEMMA 2: Consider a PBE ε^*, then, for all $T<\infty$, G^i $i=1,2$ are differentiable *a.e.* in $(0, T)$.

Proof:

Step 1: $G^1(t)=G^1(t')$ iff $G^2(t)=G^2(t')$

Assume $G^1(t)=G^1(t')$ and assume that $G^2(t)<G^2(t')$. Consider now types of player 2 that concede at some $\tau\in(t, t']$ whose payoff is $V(s_2 \mid h_\tau)=v_2(x^L-s_2)e^{-t}$. Any of these types will be better off conceding at t. Hence there cannot be any type of player 2 conceding at $\tau\in(t, t']$.

Step 2: $G^i(.)$ are absolutely continuous for all $t>0$.

Assume that for some $\delta>0$, and $t>0$, $G^1(t+\delta)-G^1(t)>\delta K$ for all $K>0$. Then any type of player 2 strictly prefers to concede at $t+\delta$ rather than at t. If we have $G^1(t+\delta')-G^1(t)\leqslant\delta' K'$ for some K', for all $\delta'<\delta$, then is necessarily the case that $G^1(t+\delta)-G^1(t+\delta')>(\delta-\delta')K$ for all $K>0$, *i.e.* 2 strictly prefers a concession at $t+\delta$ is to a concession at t or $t+\delta'$ for all $\delta'<\delta$, *i.e.* $G^2(t)=G^2(t+\delta)$, but then by step 1 we have $G^1(t)=G^1(t+\delta)$, a contradiction. If we have $G^1(t+\delta')-G^1(t)>\delta' K$ for all K, for some $\delta'<\delta$, we can repeat the previous argument for $\delta''<\delta'$, etc. This completes the proof unless $G^1(.)$ is such that $G^1(t+\delta)-G^1(t)>\delta K$ for all $K>0$ and all $\delta>0$. But this would contradict G^1's right continuity. □

LEMMA 3: At most one player makes a concession at $t=0$ with positive probability.

Proof: Assume otherwise, let P^j be the probability of j's concession at 0 and consider the players of type s_i conceding at 0. A concession at 0 yields a payoff no greater than

$$P^j v\left(\frac{x^H+x^L}{2}-z\right)+(1-P^j)v(x^L-z),$$

by delaying their first concession to some $\delta>0$, any of these types will get an expected payoff arbitraryly close to

$$P^j v(x^H-z)+(1-P^j)v(x^L-z)>P^j v\left(\frac{x^H+x^L}{2}-z\right)+(1-P^j)v(x^L-z). \quad \square$$

LEMMA 5: No Unilateral Intransigence. In any PBE, if $\mathbf{s}\ll\mathbf{x}^L$, then $\tau^{ik}(\mathbf{s})<\infty$ for all k.

Proof: The result is true for one issue. Let us assume it is true for $n-1$ issues; Assume $\tau^{ik}(\mathbf{s})=\infty$ for some $\mathbf{s}\ll\mathbf{x}^L$ for all k. Let $\alpha=\text{Prob}(\mathbf{s}_j\gg\mathbf{x}^L)$, if player i of type $\mathbf{s}$ does not concede he receives a payoff no greater than

$$\int_{[0,\infty)} \prod_{\text{all } k} v_i^k(x^{kH}-s_i^k)e^{-\tau}\,dG^j(\tau),$$

notice that for each $t>0$,

$$\int_{[0,\infty)} \prod_{\text{all } k} v_i^k(x^{kH}-s_i^k)\, e^{-\tau}\, dG^j(\tau) \leqslant \int_{[0,t)} \prod_{\text{all } k} v_i^k(x^{kH}-s_i^k)\, e^{-\tau}\, dG^j(\tau) + \prod_{\text{all } k} v_i^k(x^{kH}-s_i^k)\, e^{-t}(\alpha - G^j(t)),$$

because the right hand side is what player i can receive if he does not concede and j concentrates at t all the probability of conceding at t or later. Notice that for each $\gamma>0$, for t large enough, $\alpha - G^j(t)<\gamma$. Therefore, if $\mathbf{s} \ll \mathbf{x}^L$, then there is some $t<\infty$ such that

$$\int_{[0,t)} \prod_{\text{all } k} v_i^k(x^{kH}-s_i^k)\, e^{-\tau}\, dG^j(\tau) + \prod_{\text{all } k} v_i^k(x^{kH}-s_i^k)\, e^{-t}(\alpha - G^j(t)) < \int_{[0,t)} \prod_{\text{all } k} v_i^k(x^{kH}-s_i^k)\, e^{-\tau}\, dG^j(\tau) + \prod_{\text{all } k} v_i^k(x^{kL}-s_i^k)\, e^{-t}(1 - G^j(t)),$$

hence

$$\int_{[0,\infty)} \prod_{\text{all } k} v_i^k(x^{kH}-s_i^k)\, e^{-\tau}\, dG^j(\tau) < \int_{[0,t)} \prod_{\text{all } k} v_i^k(x^{kH}-s_i^k)\, e^{-\tau}\, dG^j(\tau) + \prod_{\text{all } k} v_i^k(x^{kL}-s_i^k)\, e^{-t}(1 - G^j(t)),$$

i.e. conceding all at t is better than not conceding in any issue.

Thus $\tau^{ik}(\mathbf{s})=T<\infty$ for some k. By the induction assumption, in the continuation after concession on issue k at T, this player has a finite date of concession for all the issues, since plans must not change as a result of own concessions, such concessions on issues other than k must have been planned at h_0, thus $\tau^{ir}(\mathbf{s})<\infty$ for all $r\neq k$ also. □

LEMMA 6: Weak Monotonicity. In any PBE, if $\mathbf{s}' \ll \mathbf{s}$, then $\tau^{ik}(\mathbf{s}')\leqslant\tau^{ik}(\mathbf{s}')$.

Proof: Let ε denote a PBE strategy profile. Let $W_{tt'}(\mathbf{z}\,|\,\varepsilon)$, denote the increase (or decrease) in expected gains that player i of type $\mathbf{z}$ faces at t when he delays concession on k from t to t'. Let

$$\Psi^{\mathbf{s}}_{tt'}(\mathbf{w}\,|\,\varepsilon) = E^{\mathbf{s}}_{tt'}(\mathbf{s}+\mathbf{w}\,|\,\varepsilon) - E^{\mathbf{s}}_{tt'}(\mathbf{s}\,|\,\varepsilon),$$

where $E^{\mathbf{x}}_{tt'}(\mathbf{z}\,|\,\varepsilon)$ denotes the expected utility to player i of type $\mathbf{z}$ obtained through the agreements that arise in $[t, t')$ when player i plays ε^i as a type $\mathbf{x}$ and player j follows ε^j. Notice that if $\mathbf{w} \gg \mathbf{0}$ ($\mathbf{w} \ll \mathbf{0}$), and ε leads to a positive probability of agreement in $[t, t')$, $\Psi^{\mathbf{s}}_{tt'}(\mathbf{w}\,|\,\varepsilon)>0$ ($\Psi^{\mathbf{s}}_{tt'}(\mathbf{w}\,|\,\varepsilon)<0$).

If $\tau^{ik}(\mathbf{s}')=t'>t=\tau^{ik}(\mathbf{s})$, we must have that $W_{tt'}(\mathbf{s}\,|\,\varepsilon^j)\leqslant 0$ and that $W_{tt'}(\mathbf{s}'\,|\,\varepsilon^j)\geqslant 0$. But this cannot hold, if a player of type $\mathbf{s}'$ does not find advantageous to delay concession k from t to t', that is $W_{tt'}(\mathbf{s}'\,|\,\varepsilon^j)\leqslant 0$, then $\mathbf{s}$ cannot find it advantageous either because his expected gains are bounded above by what he can get if he imitates $\mathbf{s}'$:

$$W_{tt'}(\mathbf{s}\,|\,\varepsilon^j) \leqslant W_{tt'}(\mathbf{s}'\,|\,\varepsilon^j) + \Psi^{\mathbf{s}'}_{tt'}(\mathbf{s}'-\mathbf{s}).$$

If ε leads to a positive probability of agreement in $[t, t')$, we get $W_{tt'}(s \mid \varepsilon^j) < 0$. Moreover, an equilibrium ε cannot lead to a zero probability of agreement in any interval $[t, t')$. Consider the type of player i whose concession leads to agreement at t'. By Lemma 5 there must be such a player. Consider the game at $h_{t'}$. Without loss of generality we call issues 1 to $r-1$ the ones whose concession has occurred at dates no later than t in $[t, t')$. The probability of j's concession on the other issues at t' must be zero (if some were to occur at t' with positive probability, any type of player i would be better off not making the concession(s) that lead to agreement until some immediate later date). Thus, conditional on h_t he gets an expected payoff

$$\prod_{k=1 \text{ to } r-1} v_i^k(x^{kH} - s_i^k) \prod_{k \geq r} v_i^k(x^{kH} - s_i^k)\, e^{-t'},$$

for some $r \geq 1$. Deviating and making concessions on issues r to n at $\underline{t}$ yields a payoff

$$\prod_{k=1 \text{ to } r-1} v_i^k(x^{kH} - s_i^k) \prod_{k \geq r} v_i^k(x^{kH} - s_i^k)\, e^{-\underline{t}}$$
$$> \prod_{k=1 \text{ to } r-1} v_i^k(x^{kH} - s_i^k) \prod_{k \geq r} v_i^k(x^{kH} - s_i^k)\, e^{-t'}. \quad \square$$

LEMMA 7: Independence of the Issues. For any PBE strategy profile ε, whether or not s_i lies in $I_i^k(t \mid \varepsilon_j)$ depends only on s_{ik}.

Proof: Pick s such that can concede only on issue 1 (*i.e.* $s_k > x_{kL}$ for all $k > 1$), then since this type will not concede on any other issue, $\tau^1(s)$ depends only on s_1.

Consider now s that can make only two concessions (*i.e.* $s_k > x_{kL}$ for all $k > 2$). Assume without loss of generality that $\tau^1(\mathbf{s}) \leq \tau^2(\mathbf{s})$. In a PBE the date to concede on issue 2 must be the same after concession on issue 1 has taken place, that is, in the subgames in which s can concede only on issue 2. By our previous paragraph, $\tau^2(\mathbf{s})$ depends only on s_2, provided that s decides to concede on 1 no later than $\tau^2(\mathbf{s})$. We will now see that $\tau^1(\mathbf{s})$ cannot depend on s_2. Assume we have s and s′ such that $s_k = s'_k$ for all $k \neq 2$, $s_2 \neq s'_2$, $s'_2 < x_{2L}$ and $\tau^1(\mathbf{s}) = t < t' = \tau^1(\mathbf{s}')$. Thus, $W_{tt'}(\mathbf{s}) \leq 0$ and $W_{tt'}(\mathbf{s}') \geq 0$, since s can imitate s' behaviour we must have

$$W_{tt'}(\mathbf{s}) \geq W_{tt'}(\mathbf{s}') + \Psi_{tt'}^{\mathbf{s}'}(\mathbf{s}' - \mathbf{s}),$$

but $\mathbf{s}' - \mathbf{s} = (0, s'_2 - s_2, 0, \ldots, 0)$, thus it must be that $s'_2 < s_2$, a contradiction to lemma 4.1.

Consider now $\mathbf{s}^1$ that can make m concessions. Assume by induction that for $k = 2$ to $m-1$ $\tau^k(\mathbf{s}^1)$, depend only on s_k. Pick $\mathbf{s}^2$ such that $s_k^1 = s_k^2$ for all $k \neq 2$, by the same argument used on the previous paragraph we have that $\tau^1(\mathbf{s}^1) = \tau^1(\mathbf{s}^2)$. Next pick $\mathbf{s}^3$ such that $s_k^2 = s_k^3$ for all $k \neq 3$, we get that $\tau^1(\mathbf{s}^1) = \tau^1(\mathbf{s}^2) = \tau^1(\mathbf{s}^3)$. Inductively we pick $\mathbf{s}^m$ such that $s_k^{m-1} = s_k^m$ for all $k \neq 3$ and we get that $\tau^1(\mathbf{s}^1) = \ldots = \tau^1(\mathbf{s}^m)$. $\square$

LEMMA 8: Consider a PBE ε, then, for all $T < \infty$, G^i and $V_i^{fk-}(s_i \mid .)$ $i = 1, 2$ are differentiable *a.e.* in $(0, T)$.

Proof:

Step 1: $G^1(t) = G^1(t')$ iff $G^2(t) = G^2(t')$.

Assume $G^1(t) = G^1(t')$ and assume that $G^2(t) < G^2(t')$. First notice that since 1 is not conceding in $[t, t']$, it cannot be the case that he concedes with positive probability in response to a concession by 2 at τ: Since concession time on issue k are independent of concession time on issue r, then concession time for issue k must be independent of whether or not a concession on issue r has taken place. Consider now types of player 2 that concede at some $\tau \in (t, t']$. It is necessarily the case that there is some $\mathbf{s}_2$ conceding at τ whose payoff is $V(\mathbf{s}_2 \mid h_\tau) = \prod_{\text{all } k} v_i^k (x^{kL} - s_i^k) e^{-(t+(n-1)\Delta)}$. But any of these types will be better off conceding at t. Hence there cannot be any type of player 2 conceding at $\tau \in (t, t']$.

Step 2: $G^i(.)$ are absolutely continuous for all $t > 0$.

Assume that for some $\delta > 0$, and $t > 0$, $G^1(t+\delta) - G^1(t) > \delta K$ for all $K > 0$. Then any type of player 2 strictly prefers to concede at $t+\delta$ rather than at t. If we have $G^1(t+\delta') - G^1(t) \leqslant \delta' K'$ for some K', for all $\delta' < \delta$, then is necessarily the case that $G^1(t+\delta) - G^1(t+\delta') > (\delta - \delta')K$ for all $K > 0$, *i.e.* 2 strictly prefers a concession at $t+\delta$ is to a concession at t or $t+\delta'$ for all $\delta' < \delta$, *i.e.* $G^2(t) = G^2(t+\delta)$, but then by step 1 we have $G^1(t) = G^1(t+\delta)$, a contradiction. If we have $G^1(t+\delta') - G^1(t) > \delta' K$ for all K, for some $\delta' < \delta$, we can repeat the previous argument for $\delta'' < \delta'$, etc. This completes the proof unless $G^1(.)$ is such that $G^1(t+\delta) - G^1(t) > \delta K$ for all $K > 0$ and all $\delta > 0$. But this would contradict G^1's right continuity. □

LEMMA 9: At most one player makes a concession at $t = 0$ with positive probability.

Proof: Assume otherwise, let P^j be the probability of j's concession at 0 and consider the players of type $\mathbf{s}_i$ conceding at 0. It is necessarily the case that there is some $\mathbf{s}_2$ conceding on all issues at 0 whose payoff is no more than

$$P^j v\left(\frac{x^{rH} + x^{rL}}{2} - s_i^r\right) \prod_{\text{all } k \neq r} v_i^k (x^{kL} - s_i^k) + (1 - P^j) \prod_{\text{all } k} v_i^k (x^{kL} - s_i^k),$$

By delaying their total concession to some $\delta > 0$, any of these types will get an expected payoff arbitraryly close to

$$P^j v((x^{rH} - s_i^r) \prod_{\text{all } k \neq r} v_i^k (x^{kL} - s_i^k) + (1 - P^j) \prod_{\text{all } k} v_i^k (x^{kL} - s_i^k)$$
$$> P^j v\left(\frac{x^{rH} + x^{rL}}{2} - s_i^r\right) \prod_{\text{all } k \neq r} v_i^k (x^{kL} - s_i^k) + (1 - P^j) \prod_{\text{all } k} v_i^k (x^{kL} - s_i^k). \quad \square$$

• References

AUSUBEL, L. M. and DENECKERE , R. J. (1988 *a*). – "Stationary Sequential Equilibria in Bargaining with Two-Sided Incomplete Information", Center for Mathematical Studies in Economics and Management Science Working Paper.

AUSUBEL, L. M. and DENECKERE, R. J. (1988 *b*). – "Efficient Sequential Bargaining", Center for Mathematical Studies in Economics and Management Science Working Paper.

CRAMTON, P. C. (1984 *b*). – "Bargaining with Incomplete Information: An Infinite Horizon Model with Continuous Uncertainty", *Review of Economic Studies*, 51, pp. 579-594.

CRAMPTON, P. C. (1987). – "Strategic Delay in Bargaining with Two-Sided Uncertainty", Yale School of Organization and Management Working Paper 36.

CHATTERJEE, K and SAMUELSON, W. F. (1988). – "Bargaining with Two-sided Incomplete Information", *Review of Economic Studies*, 54, pp. 175-192.

CHO, I. (1988). – "Uncertainty and Delay in Bargaining", University of Chicago, Department of Economics Working Paper, 88-63.

FUDENBERG, D., LEVINE, D. and TIROLE, J. (1985). – "Infinite Horizon Models of Bargaining with Incomplete Information", in Roth, A.E. (ed.), *Game Theoretic Models of Bargaining*, Cambridge University Press, pp. 73-88.

FUDENBERG, D. and TIROLE, J. (1983). – "Sequential Bargaining under Incomplete Information", *Review of Economic Studies*, 50, pp. 221-248.

GROSSMAN, S. and PERRY, M. (1986). – "Sequential Bargaining under Asymmetric Information", *Journal of Economic Theory*, 39, pp. 120-154.

GUL, F. and SONNENSCHEIN, H. (1988). – "On Delay in Bargaining with One-Sided Uncertainty", *Econometrica*, 56, pp. 601-612.

GUL, F. SONNENSCHEIN, H. and WILSON, R. B. (1986). – "Fondations of Dynamic Monopoly and the Coase Conjecture", *Journal of Economic Theory*, 39, pp. 155-190.

HENDRICKS, K. WEISS, A. and WILSON, C. (1987). – "The War of Attrition in Continuous Time with Complete information", *Mimeo*.

KREPS, D. M. and WILSON, R. B. (1982). – "Reputation and Imperfect Information", *Journal of Economic Theory*, 27, pp. 553-579.

LINHART, P. B. and RADNER, R. (1988). – "Minimax - Regret Strategies for Bargaining Over Several Variables", *Mimeo*.

PERRY, M. (1986). – "An Example of Price Formation in Bilateral Situations: a Bargaining Model with Incomplete Information", *Econometrica*, 54, pp. 313-322.

PONSATI, C. (1989). – "Two- Sided Incomplete Information Bargaining with a Finite Set of Agreements", *Mimeo*.

RUBINSTEIN, A. (1982). – "Perfect Equilibrium in a Bargaining Model", *Econometrica*, 50, pp. 97-109.

RUBINSTEIN, A. (1985). – "A Bargaining Model with Incomplete Information". *Econometrica*, 53, pp. 1151-1172.

SOBEL, J. and TAKAHASHI, I. (1983). – "A Multistage Model of Bargaining", *Review of Economic Studies*, 50, pp. 411-426.