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The Assignment Game: The Reduced Game

Guillermo OWEN *

ABSTRACT. – Let v be an assignment game. For a given reference payoff vector $(\mathbf{x}; \mathbf{y})$, and a coalition S, bargaining within the coalition can be represented by either the reduced game or the derived game. It is known that the reduced game need not be an assignment game (in fact, it need not be superadditive) while the derived game is another assignment game, with modified reservation prices.

We prove that, when the reference vector is in the core of the game, the derived game is the superadditive cover of the reduced game.

Jeu d'assignation : le jeu réduit

RÉSUMÉ. – Soit v un jeu d'assignation. Étant donné un vecteur de paiement $(\mathbf{x}; \mathbf{y})$ et une coalition S, on peut représenter une négociation à l'intérieur d'une coalition comme un jeu réduit ou comme un jeu dérivé. On sait que le jeu réduit n'est pas nécessairement un jeu d'assignation (en fait il n'est pas nécessairement sur-additif) alors que le jeu dérivé reste, avec des valeurs de réserve différentes, un jeu d'assignation. On montre que si le vecteur de référence est dans le cœur du jeu, le jeu dérivé est la couverture sur-additive du jeu réduit.

* G. OWEN: Department of Mathematics, Naval Postgraduate School, Monterey, CA 93943, USA. Support from the University of Caen, France, is gratefully acknowledged.

1 Introduction

The assignment problem is a well-known linear programming problem; it appears in the literature at least as early as DWYER [1954]. Originally it was studied as a (one-sided) optimization problem; however, it was soon clear that this was (at least under some circumstances) better considered as a multi-person problem. Assignment games (also known as marriage games or matching games) were first specifically studied in SHAPLEY and SHUBIK [1971] where it was shown that the core of the game coincides with the set of all optimal vectors for the dual linear program. Since then a large literature has developed. An important problem deals with selection of one of the core payoffs when (as frequently happens) the core contains more than one point. (*See* ROCHFORD [1984], ROTH [1984] for this, and especially ROTH, SOTOMAYOR [1990] which gives a very complete treatment of the problem.) One approach to this selection deals with the reduced game as defined in PELEG [1986], which has also proved aplicable to NTU games (*see* MOLDOVANU [1990]). This is the approach which we will use below.

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2 Assignment Games

Let J and K be disjoint finite sets. An assignment from J to K is a 1-1 function α with domain

$$\operatorname{dom} \alpha \subset J$$

and image

$$\operatorname{Im} \alpha \subset K.$$

An assignment game is a quintuple (M, N, A, **p**, **q**), where M and N are disjoint finite sets (the two sides of the market) of cardinality m and n respectively, A is an $m \times n$ matrix (the marriage incomes), and **p** and **q** are an m-vector and an n-vector respectively (the reservation prices or celibate incomes).

For any $J \subset M$, $K \subset N$, and any assignment α from J to K, we define

(1) $$w(\alpha; J, K) = \sum_{j \in \operatorname{dom}(\alpha)} a_{j,\alpha(j)} + \sum_{j \in J - \operatorname{dom}(\alpha)} p_j + \sum_{k \in K - \operatorname{Im}(\alpha)} q_k.$$

We can now define a game v in characteristic function form, with player set $L = M \cup N$, by

$$v(S) = \max_{\alpha} w(\alpha; S \cap M, S \cap N) \tag{2}$$

where the maximum is taken over all assignments α from $S \cap M$ to $S \cap N$.

It is easy to see that this game v is superadditive. Moreover, the only effective coalitions for this game are (*a*) the one-person coalitions, (*b*) coalitions $\{j, k\}$ where $j \in M$, $k \in N$.

As is well known (*see*, *e.g.*, GASS [1958]), the problem of choosing an optimal assignment from J to K (*i.e.*, an assignment α which maximizes $w(\alpha; J, K)$) can be expressed as a linear program. The dual linear program consists in minimizing the sum

$$x(J) + y(K) = \sum_{j \in J} x_j + \sum_{k \in K} y_k \tag{3}$$

subject to

$$x_j + y_k \geq a_{jk} \quad \text{for all } j, k \tag{4}$$

$$x_j \geq p_j \quad \text{for all } j \tag{5}$$

$$y_k \geq q_k \quad \text{for all } k. \tag{6}$$

By the duality theorem of linear programming, we know that if α^* is an optimal assignment from J to K, and $(\mathbf{x}^*, \mathbf{y}^*)$ is a minimizing vector for the dual program, then

$$w(\alpha^*; J, K) = x^*(J) + y^*(K) \tag{7}$$

and, by the complementary slackness theorem,

$$x_j^* + y_k^* = a_{jk} \qquad \text{if} \quad k = \alpha^*(j) \tag{8}$$

$$x_j^* = p_j \qquad \text{if} \quad j \in J\text{-dom}(\alpha^*) \tag{9}$$

$$y_k^* = q_k \qquad \text{if} \quad k \in K\text{-Im}(\alpha^*). \tag{10}$$

It will follow from this that if $(\mathbf{x}^*, \mathbf{y}^*)$ is an optimal dual vector for the assignment from M into N, then $(\mathbf{x}^*, \mathbf{y}^*)$ is in the core of the game v, and conversely.

Interpretation: There are several interpretations; the reader is referred to SHAPLEY and SHUBIK [1972] for a fuller discussion.

In our case, we interpret M and N to be the sets of men and women respectively; a_{jk} is the income that man j and woman k will receive if married; p_j and q_k are the incomes that they would receive by remaining single.

An assignment α is a set of marriages; the 1-1 property guarantees that each man marries at most one woman, and each woman marries at most one man. An imputation is an allotment of incomes to the $m+n$ individuals. If α^* is an optimal assignment from M to N, and $(\mathbf{x}^*, \mathbf{y}^*)$ is in the core of the game, then condition (8) tells us that each married couple or unmarried individual receives exactly its/his/her income; conditions (4-6) guarantee that the assignment is stable, *i.e.*, there is no man who can gain by leaving his marriage to become single, no woman who can gain by leaving her marriage to become single, and no pair (man and woman) who can gain by marrying each other (previously divorcing their current mates if applicable).

3 The Derived and Reduced Games

Suppose some subset S of L is given. The members of S might choose to play a game among themselves. If so, there are two possibilities. The first – and simpler – is that they effectively forget about the existence of the remaining players, L − S. (This might happen if, *e.g.*, S and L − S are distinct castes in a society where intermarriage is strictly forbidden.) In such a case, the members of S will play the restricted game, $v|_S$, among themselves, defined by

$$v|_S(T) = v(T)$$

for $T \subset S$. There is little to say about this.

Alternatively, the members of S might bargain among themselves while taking into account the existence of L − S, *i.e.* some $i \in S$ will be strengthened if he (or she) can do well by joining some $h \in L - S$. This, however, depends on what h will "charge" for her (his) cooperation, which in turn depends on how well h is currently doing. Thus, in this case, the game among the members of S should take some current payoff vector (**x**; **y**) into account. We consider two possible ways of doing this: the reduced game, and the derived game.

The reduced game, as defined in Peleg [1986], assumes a payoff vector **z** is given. For a fixed $S \subset L$, we defined a game with player set S by

(11) $$v^*_{S,z}(T) = \max_{H \subset L-S} \{v(T \cup H) - \sum_{i \in H} z_i\}$$

if $T \subset S$, $T \neq S$, $T \neq \varphi$. For $T = \varphi$ or S, we have

(12) $$v^*_{S,z}(\varphi) = 0$$

and

(13) $$v^*_{S,z}(S) = \sum_{j \in S} z_j.$$

For this particular game, we will have $\mathbf{z}=(\mathbf{x};\ \mathbf{y})$. Thus the reduced game function takes the form

(14) $$v^*_{S,x,y}(T)=\max_{H\subset L-S}\left\{v(T\cup H)-\sum_{j\in H\cap M}x_j-\sum_{k\in H\cap N}y_k\right\}.$$

As an alternative to this, we may consider the derived game. Essentially, the derived game $\hat{v}_{S,x,y}$ is simply another assignment game, with player set S, and defined by the quintuple $(S\cap M,\ S\cap N,\ \hat{A},\ \hat{\mathbf{p}},\ \hat{\mathbf{q}})$, where

(15) $\hat{A}$ is the submatrix of A, corresponding to rows in $S\cap M$, and to columns in $S\cap N$;

(16) $$\hat{p}_j=\max\left\{p_j,\ \max_{i\in N-S}(a_{ji}-y_i)\right\}$$

(17) $$\hat{q}_k=\max\left\{q_k,\ \max_{h\in M-S}(a_{hk}-x_h)\right\}.$$

Essentially, in the derived game, the members of S consider the two alternatives: (*a*) marrying within S, and (*b*) not marrying within S. If they do marry within S, the marriage incomes in $\hat{v}$ are the same as in the original game. If, however, some member of S chooses not to marry within S, he has the choice of either remaining single (thus getting the original celibate income p_j or q_k), or else of marrying some member of $L-S$, which leaves an income $a_{ji}-y_i$ or $a_{hk}-x_h$ after the prospective spouse is given y_i or x_h. Presumably, he (or she) would choose whatever is best for him (or her) in this case to obtain a new reservation price.

In general, the two games v^*_S and $\hat{v}_S$ are not equal. This is easy to see because, for example, v^* need not be superadditive, while $\hat{v}$, as an assignment game, is always superadditive.

To see that v^* need not be superadditive, consider the 3×3 assignment game given by $M=\{1,2,3\}$, $N=\{4,5,6\}$,

$$A=\begin{array}{c}\\1\\2\\3\end{array}\begin{array}{c}\begin{array}{ccc}4&5&6\end{array}\\\begin{pmatrix}2&2&2\\2&2&2\\2&2&2\end{pmatrix}\end{array}$$

and $\mathbf{p}=\mathbf{0}$, $\mathbf{q}=\mathbf{0}$.

We choose $(\mathbf{x};\ \mathbf{y})=(1, 1, 1;\ 1, 1, 1)$ (which is the nucleolus of the game) and set $S=\{1, 2, 4, 5\}$. Then

$$v^*_S(\{1\})=\max_{H\subset\{3,6\}}\left\{v(H\cup\{1\})-\sum_{i\in H}z_i\right\}$$

and this maximum is obtained by seting $H=\{6\}$, so that

$$v^*_S(\{1\})=1.$$

By symmetry, we will also have

$$v^*_S(\{2\})=1.$$

Similarly,

$$v_S^*(\{1, 2\}) = \max_{H \subset \{3, 6\}} \{v(H \cup \{1, 2\}) - \sum_{i \in H} z_i\}$$

and the maximum here, also obtained by setting $H = \{6\}$, gives

$$v_S^*(\{1, 2\}) = 1.$$

Thus $v^*(\{1, 2\}) < v^*(\{1\}) + v^*(\{2\})$, and v^* is not superadditive.

4 Relationship between the Reduced and Derived Games

The question, then, remains as to the relationship between the reduced and derived games, v^* and $\hat{v}$. For this we need a further definition.

DEFINITION: Let u be a game (not necessarily superadditive) on player set L. The superadditive cover of u is the minimal superadditive game w such that $w \geq u$.

It is not obvious, from the definition, that such a minimal w exists. However, the game w given by

(18) $$w(S) = \max \sum_l u(T_l)$$

where the maximum is taken over all partitions $\{T_1, \ldots, T_r\}$ of S, is the desired game.

We are now in a position to state our main result:

THEOREM: If the imputation $(\mathbf{x}; \mathbf{y})$ lies in the core of the assignment game v, then, for any S, the derived game $\hat{v}_{S, x, y}$ is the superadditive cover of the reduced game $v^*_{S, x, y}$.

Proof: We must prove (*a*) $\hat{v}$ is superadditive; (*b*) $\hat{v} \geq v^*$; (*c*) $\hat{v}$ is minimal.

(*a*) It is clear that $\hat{v}$ is superadditive as all assignment games are superadditive.

(*b*) For $T \subset S$, there exists some $H \subset L - S$ such that

$$v^*(T) = v(T \cup H) - \sum_{j \in H \cap M} x_j - \sum_{k \in H \cap N} y_k.$$

Moreover, $v(T \cup H) = w(\alpha; J, K)$ for some α from J to K, where $J = (T \cup H) \cap M$, and $K = (T \cup H) \cap N$.

Now, α partitions the set $T \cup H$ into six classes :

(1) marriages within T (*i.e.*, $j, k \in T$, $\alpha(j)=k$)

(2) marriages within H ($j, k \in H$, $\alpha(j)=k$)

(3) marriages between men in T, women in H ($j \in T$, $\alpha(j)=k \in H$)

(4) marriages between men in H, women in T ($j \in H$, $\alpha(j)=k \in T$)

(5) single members of T ($j \in T \cap M - \text{dom}(\alpha)$ or $k \in T \cap N - \text{Im}(\alpha)$)

(6) single members of H ($j \in H \cap M - \text{dom}(\alpha)$ or $k \in H \cap N - \text{Im}(\alpha)$).

Each of these sets, moreover, can be partitioned into men and women, *i.e.* members of J (or M) and members of K (or N), giving us the 12 sets $J_1, \ldots, J_6, K_1, \ldots, K_6$.

Now,

$$w(\alpha; J, K)=\sum_{1,2,3,4} a_{jk}+\sum_{J_{5,6}} p_j+\sum_{K_{5,6}} q_k \tag{20}$$

and so

$$v^*(T)=\sum_{1,2,3,4} a_{jk}+\sum_{J_{5,6}} p_j+\sum_{K_{5,6}} q_k-\sum_{J_{2,4,6}} x_j-\sum_{K_{2,3,6}} y_k. \tag{21}$$

We will rewrite this in the form

$$\begin{aligned} v^*(T)=&\sum_1 a_{kj}+\sum_2 (a_{jk}-x_j-y_k) \\ &+\sum_3 (a_{jk}-y_k)+\sum_4 (a_{jk}-x_j)+\sum_{J_5} p_j+\sum_{K_5} q_k+\sum_{J_6} (p_j-x_j)+\sum_{K_6} (q_k-y_k). \end{aligned} \tag{22}$$

Now $(\mathbf{x}; \mathbf{y})$ is in the core of v, and so $x_j \geqslant p_j$, $y_k \geqslant q_k$, and $x_j+y_k \geqslant a_{jk}$. Thus the sums labeled 2 and 6 in equation (22) are non-positive, and so

$$v^*(T) \leqslant \sum_1 a_{jk}+\sum_3 (a_{jk}-y_k)+\sum_4 (a_{jk}-x_j)+\sum_{J_5} p_j+\sum_{K_5} q_k. \tag{23}$$

Also, by (16)-(17), $a_{jk}-y_k \leqslant \hat{p}_j$, $a_{jk}-x_j \leqslant \hat{q}_k$, $p_j \leqslant \hat{p}_j$, and $q_k \leqslant \hat{q}_k$. Thus,

$$v^*(T) \leqslant \sum_1 a_{jk}+\sum_{J_{3,5}} \hat{p}_j+\sum_{K_{4,5}} \hat{q}_k. \tag{24}$$

But the right side of (24) is precisely $\hat{w}(\beta; T \cap M, T \cap N)$, where β is the restriction of α to $T \cap M$, and where the circumflex accent on w means that reservation prices $\hat{p}_j$ and $\hat{q}_k$ are used (rather than p_j and q_k). Moreover,

$$\hat{v}(T)=\max_\tau \{ \hat{w}(\tau; T \cap M, T \cap N) \} \tag{25}$$

and so

$$v^*(T) \leqslant \hat{v}(T) \tag{26}$$

as desired.

(*c*) Minimality. To prove this, consider first the subset $T=\{j\}$, where $j \in S \cap M$. We have

(27) $$v^*(\{j\}) = \max_{H \subset L-S} \{v(H \cup \{j\}) - \sum_{i \in H} z_i\}$$

and this maximum will be at least as much as if only one-player sets $\{k\}$, $k \in N-S$, are considered:

$$v^*(\{j\}) \geqslant \max_{k \in N-S} \{v(\{j, k\}) - y_k\}.$$

Now $v(\{j, k\}) \geqslant a_{jk}$, and so

$$v^*(\{j\}) \geqslant \max_{k \in N-S} \{a_{jk} - y_k\}.$$

We could also consider $H = \varphi$, giving

$$v^*(\{j\}) \geqslant v(\{j\}) = p_{j'},$$

and, by definition of $\hat{p}_j$, we see

(28) $$v^*(\{j\}) \geqslant \hat{p}_j.$$

By symmetry, we will have, for $k \in S \cap N$,

(29) $$v^*(\{k\}) \geqslant \hat{q}_k.$$

Next, let $T=\{j, k\}$, where $j \in S \cap M$ and $k \in S \cap N$. We have

$$v^*(\{j, k\}) = \max_{H} \{v(H \cup \{j, k\}) - \sum_{i \in H} z_i\}$$

and, letting $H = \varphi$, this gives us

$$v^*(\{j, k\}) \geqslant v(\{j, k\})$$

or

(30) $$v^*(\{j, k\}) \geqslant a_{jk}.$$

It follows that, if u is a superadditive game with $u \geqslant v^*$, then for any T we must have

(31) $$u(T) \geqslant \sum a_{jk} + \sum p_j + \sum q_k$$

for any partition of T into one- and two-player sets. But this means $u \geqslant \hat{v}$, and we conclude that $\hat{v}$ is minimal. Thus $\hat{v}$ is the superadditive cover of v^*.

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