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The Fictitious Payoff-Function:

Two Applications to Dynamic Games

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ABSTRACT. – The notion of a fictitious-payoff function is developed in the context of a dynamic model of oligopoly. It is shown that in a certain class of games, the oligopolistic market acts as if it were maximizing a single objective function – the fictitious-payoff function. The more complex problem of calculating multiplayer subgame-perfect equilibria is thus reduced to an ordinary optimization problem.

La fonction de gain fictive : deux applications aux jeux dynamiques

RÉSUMÉ. – On développe un concept de fonction de gain fictive dans le contexte d'un modèle dynamique d'oligopole. On montre que, pour une certaine classe de jeux, le marché oligopolistique agit comme s'il maximisait une fonction objectif – la fonction de gain fictive –. Le porblème complexe du calcul des équilibres parfaits d'un jeu à plusieurs joueurs se réduit ainsi à un problème de maximisation ordinaire.

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1 Introduction

The notion of a fictitious-payoff function is developed in the context of a dynamic model of oligopoly. It is shown that in a certain class of games, the oligopolistic market behaves as if it were maximizing a single objective function – the fictitious-payoff function. The more complex problem of calculating multiplayer subgame-perfect equilibria is thus reduced to an ordinary optimization problem.

The idea that an oligopolistic market is observationally equivalent to one where a single agent maximizes a fictitious-objective function is similar to the notion that a competitive market is indistinguishable from one in which a planner maximizes a social-welfare function. The optimization approach to solving competitive-market equilibria, which was pioneered by Negishi [1960], has proved useful in the solution of large-scale models (Dixon [1975], and Ginsburgh and Waelbroeck [1976]).

In section 2 of this paper, necessary and sufficient conditions for the existence of a fictitious-payoff or oligopoly-market-objective function are given. Examples are then presented to clarify the restrictions on the individual-player-profit functions which must be satisfied if a market-objective function is to exist. In particular, it is shown that the restrictions on the structure of demand derived by Spence [1976], who shows that monopolistically competitive firms maximize the "wrong" surplus function, are a special case of those derived here.

In sections 3 and 4, the fictitious-payoff function is put to work. The first application is graphical. Here the fictitious-payoff function is used to illustrate strategic behavior in dynamic games. The example is one of learning by doing, where firm costs fall with cumulative production. The multiplant-monopolist's and surplus-maximizer's objective functions are compared to the fictitious-payoff functions for open and closed-loop (subgame-perfect) solutions to the game. In addition, it is demonstrated that in this model of private learning by doing, the difference between closed and open-loop objective functions does not disappear as the number of players grows.

The second application is numerical. Here the fictitious-payoff function is used to calculate Nash equilibria of large models. First, a method of finding all equilibria of the game is sketched. And second, the fictitious-payoff method of computing Nash equilibria is compared to other computational algorithms. The fictitious-payoff technique is closest in spirit to Rosen's [1965] fictitious gradient, but it requires no modifications to common mathematical-programming software. Imposition of the restrictions required for a fictitious-payoff function to exist can be a small price to pay for the computational simplicity that is gained. In particular, solving large multi-sector models with imperfect competition can be considerably simplified.

The final section of the paper summarizes and concludes. The proof of the main proposition is found in the appendix.

2 The Fictitious-Payoff Function

Consider a game with N players where each player seeks to maximize his own gain. A strategy for player i is a choice of a number x^i from some compact subset $X=[\underline{x}, \bar{x}]$ of the real line. Let x be the vector of N choices x^i, $i=1, \ldots, N$. The payoff for player i is a function $\pi^i(x)$ which maps any vector x into the real line. Payoff functions are assumed to possess all first and second partial derivatives. Usually players will be called firms and the i-th player's payoff will be that firm's profit or discounted-profit stream. The choice variable x^i can be price, quantity, advertising, capacity, or any other variable under firm i's control.

The equilibrium concept of interest is the Nash equilibrium, which is the solution to

$$(1) \qquad \max_{x^i} \pi^i(x), \qquad i=1, \ldots, N,$$

conditional on $x^{-i}:=(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^N)'$.[1] Let x^* denote this solution.

More generally, consider a two-period game.[2] In each period players choose x_t^i, $t=1, 2$, from a compact set X_t. Firm i's profit in the first [second] period is $\pi_1^i(x_1)$ $[\pi_2^i(x_1, x_2)]$ and its payoff function is $\Pi^i(x_1, x_2)$,

$$(2) \qquad \Pi^i(x_1, x_2):=\pi_1^i(x_1)+\delta\pi_2^i(x_1, x_2),$$

where δ is the discount factor. Without loss of generality, δ is set equal to one.

x_1 enters the second-period profit functions indicating that payoffs can differ by period for either exogenous or endogenous reasons. Endogenous differences occur when the previous-period-x choices shift the current-period-payoff functions. For example, players might choose capacity in the first period and output in the second. In this case, first-period-capacity choices alter continuation-game-profit functions.

1. The symbol ′ denotes the transpose of a (column) vector.
2. The modifications required for M periods are straight forward as long as M is finite.

With this dynamic game, two solution concepts are of interest. The first is the open-loop equilibrium, which is the solution to

$$(3) \qquad \max_{x_1^i, x_2^i} \Pi^i(x_1, x_2), \qquad i=1, \ldots, N,$$

conditional on x_1^{-i} and x_2^{-i}. Let $x^{OL}=(x_1^{OL'}, x_2^{OL'})'$ denote this solution. x^{OL} is a 2 N vector.

The open-loop equilibrium can be contrasted to the (subgame-perfect) closed-loop equilibrium, which can be calculated sequentially as follows. In the second period, each player chooses x_2^i to maximize

$$(4) \qquad \max_{x_2^i} \pi_2^i(x_1, x_2), \qquad i=1, \ldots, N,$$

conditional on x_1 and x_2^{-i}. Let the solution to equations 4 be $x_2^{CL}(x_1)$ and corresponding profits be

$$(5) \qquad \pi_2^{i*}(x_1) := \pi_2^i[x_1, x_2^{CL}(x_1)].$$

π_2^{i*} is clearly a function only of the first-period-choice variables.

In the first period, players maximize

$$(6) \qquad \max_{x_1^i} \pi_1^i(x_1) + \pi_2^{i*}(x_1).$$

Finally, let the solution to (6) be x_1^{CL}. The closed-loop equilibrium x^{CL} is then $(x_1^{CL'}, x_2^{CL}(x_1^{CL})')'$.

It is now time to introduce the fictitious-payoff function. Although the strategy spaces differ, the maximizations (1), (3), and (6) are all examples of the general problem of calculating Nash equilibria of games. (1), (3), (6) can each be written as

$$(7) \qquad \max_{x^i} \Psi^i(x), \qquad i=1, \ldots, n,$$

conditional on x^{-i}.[3] First-order conditions for the maximization of (7) are

$$(8) \qquad \partial\Psi^i(x)/\partial x^i = 0, \qquad i=1, \ldots, n.$$

The question is, under what conditions does there exist a function F (x) with the property that

$$(9) \qquad \partial F(x)/\partial x^i = \partial\Psi^i(x)/\partial x^i, \qquad i=1, \ldots, n.$$ [4]

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3. The fact that one player may have more than a single choice variable (as in maximization (3) makes no difference. One must still solve n first-order conditions for the Nash equilibrium. By Young's theorem, it will always be true that $\Psi_{ij}^i = \Psi_{ji}^i$. Therefore, no new integrability problems are introduced by this modification.

4. The answer to this question can be found in SLADE [1988 *a*]. As the proof is both simple and intuitive, it is reproduced in the appendix.

When the function $F(x)$ exists, we call it the fictitious-payoff function. In this case, the game with N players and N payoff functions is observationally equivalent to a problem where a single agent maximizes F.

Clearly, $F(x)$ exists if and only if the n first-order conditions (8) are integrable. Necessary and sufficient conditions for integrability are given by

PROPOSITION 1: Given a game with payoff functions that possess first and second-partial derivatives, the fictitious-payoff function $F(x)$ exists if and only if the individual payoff functions $\Psi^i(x)$ can be written as

$$\Psi^i(x) = \Psi(x) + \Omega^i(x^i) + \Theta^i(x^{-i}), \qquad i = 1, \ldots, n. \tag{10}$$

When (10) is satisfied, the fictitious-payoff function is

$$F(x) = \Psi(x) + \sum_i \Omega^i(x^i). \tag{11}$$

Proof: See the appendix.

Proposition 1 says that the players' payoff functions must consist of three additively separable parts: one which is common to all players, one which is specific to firm i and involves only i's strategic variable, and one which is specific to firm i and involves only the strategic variables of i's rivals.

When players are oligopolists, it is informative to contrast the fictitious-payoff function to the objective function of a monopolist who controls all N plants. Call this function $G(x)$, where

$$G(x) := \sum_i \Psi^i(x) = N\,\Psi(x) + \sum_i \Omega^i(x^i) + \sum_i \Theta^i(x^{-i}). \tag{12}$$

The oligopoly-objective function counts the common part, $\Psi(x)$, only once whereas the monopolist counts it N times, once for each plant. In addition, the oligopoly gives the $\Theta^i(x^{-i})$ terms zero weight. Both of these differences stem from the fact that each oligopolist ignores the profits and losses that it inflicts on others.

A final point concerns a symmetric game (one with symmetric payoffs) which also satisfies restrictions (10). When all players choose the same $x^i = z$, $i = 1, \ldots, N$, define $\psi(z) := \Psi^i(z, z, \ldots, z)$, $f(z) := F(z, z, \ldots, z)$, and $g(z) := G(z, z, \ldots, z)$. The symmetric firm-objective function $\psi(z)$ has the same maximum as the symmetric monopoly-objective function $g(z)$. The common maximum corresponds to the vector of joint-profit-maximizing choice variables. This is not true, however, of the symmetric fictitious-objective function $f(z)$ which is maximized at the symmetric-Nash equilibrium.

Some examples may clarify the restrictions implied by proposition one. First consider a static three-player homogeneous-product Cournot-Nash game. [5] Firms choose output q^i, which costs $C^i(q^i)$ to produce. In

5. BERGSTROM and VARIAN [1985] consider this problem and claim (erroneously) that we must restrict attention to symmetric games.

this game, the fictitious-payoff function F exists if and only if firm i's profit function can be written as

$$\begin{aligned}(13)\qquad \pi^i(q) &= [a - b(q^1 + q^2 + q^3)]q^i - C^i(q^i) \\ &= b(-q^1 q^2 - q^1 q^3 - q^2 q^3) && (\Psi(q)) \\ &\quad + (a - bq^i)q^i - C^i(q^i) && (\Omega^i(q^i)) \\ &\quad + bq^j q^k, \quad j \neq i, \quad k \neq i, \quad j \neq k, && (\Theta^i(q^{-i})),\end{aligned}$$

Here F equals

$$(14)\qquad F(q) = -b(q^1 q^2 + q^1 q^3 + q^2 q^3) + \sum_i [(a - bq^i)q^i - C^i(q^i)].$$

In this example, the only restriction on the general Cournot-Nash problem is that demand be linear. Cost functions C^i are unrestricted by proposition 1. [6]

Next, consider a differentiated-product industry as in SPENCE [1976]. For simplicity of notation, the case of $N = 3$ firms is again considered. In the Spence model, the i'th firm's profit function is

$$\begin{aligned}(15)\qquad \pi^i(q)/\alpha_i &= h^i[(q^i)^{\alpha i}] + h^{ij}[(q^i)^{\alpha i}(q^j)^{\alpha j}] + h^{ik}[(q^i)^{\alpha i}(q^k)^{\alpha k}] \\ &\quad + h^{ijk}[(q^i)^{\alpha i}(q^j)^{\alpha j}(q^k)^{\alpha k}] - [C^i(q^i) + F^i]/\alpha_i \\ &= \{h^{ij}[(q^i)^{\alpha i}(q^j)^{\alpha j}] + h^{ik}[(q^i)^{\alpha i}(q^k)^{\alpha k}] + h^{jk}[(q^j)^{\alpha j}(q^k)^{\alpha k}] \\ &\quad + h^{ijk}[(q^i)^{\alpha i}(q^j)^{\alpha j}(q^k)^{\alpha k}]\} && (\Psi(q)) \\ &\quad + h^i[(q^i)^{\alpha i}] - [C^i(q^i) + F^i]/\alpha_i && (\Omega^i(q^i)) \\ &\quad - h^{jk}[(q^j)^{\alpha j}(q^k)^{\alpha k})], \quad j \neq i, \quad k \neq i, && (\Theta^i(q^{-i})),\end{aligned}$$

where the h's are arbitrary functions of a scalar and the α's are positive constants. [7] (15) shows that Spence's profit functions satisfy the restrictions of proposition 1 and that therefore the fictitious-payoff function exists, Here F equals

$$\begin{aligned}(16)\quad F(q) &= h^{123}[(q^1)^{\alpha 1}(q^2)^{\alpha 2}(q^3)^{\alpha 3}] \\ &\quad + h^{12}[(q^1)^{\alpha 1}(q^2)^{\alpha 2}] + h^{13}[(q^1)^{\alpha 1}(q^3)^{\alpha 3}] \\ &\quad + h^{23}[(q^2)^{\alpha 2}(q^3)^{\alpha 3}] + \sum_i \{h^i[(q^i)^{\alpha i}] - [C^i(q^i) + F^i]/\alpha_i\},\end{aligned}$$

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6. With price competition, in contrast, there are restrictions on the cost function. For example, suppose that each firm produces a differentiated product q^i with demand function $q^i = f^i(p)$. The i'th firm's profit is then

$$\pi^i(p) = f^i(p)p^i - C^i(f^i(p)).$$

In this case, rival strategic variables enter the cost function, implying that its functional form must be restricted.

7. In Spence's notation, $h(y) = H'(y)y$. Maximizing π^i/α_i is clearly equivalent to maximizing π^i.

which is Spence's "wrong" surplus function. This example shows that in the differentiated-product case, demand need not be linear.

With the two examples just discussed, the additive separability required for the existence of a fictitious-payoff function is superimposed on the additive separability between revenues and costs. This makes the conditions embodied in (10) seem more restrictive than they in fact are. For the final example, we consider general net-revenue functions for multi-output firms. Assume that there are two firms and that the first (second) produces a vector of products $x(y)$. [8] In addition, assume that firms use a vector of inputs which they purchase in competitive-factor markets at parametric prices v. The i-th firm's net-revenue function (revenues minus costs) is $R^i(x, y, v)$.

As is common in empirical applications, we work with a second-order approximation to $R^i(x, y, v)$, a flexible-functional form. For example, the following is a variant of the Generalized-Leontief function (DIEWERT [1971]),

$$(17) \quad R^i(x, y, v) \cong \sum_j \alpha^i_j v_j + \sum_k \beta^i_k x_k + \sum_m \nu^i_m y_m + \sum_j \sum_k \gamma^i_{jk} (v_j)^{1/2} (v_k)^{1/2} + \sum_j \sum_k \omega^i_{jk} (v_j)^{1/2} (x_k)^{1/2} + \sum_j \sum_k \eta^i_{jk} (v_j)^{1/2} (y_k)^{1/2} + \sum_k \sum_n \delta^i_{kn} (x_k)^{1/2} (y_n)^{1/2},$$

where Greek letters are parameters. A slight modification of (17) would produce the normalized-quadratic function (LAU [1976]).

Clearly (17) must satisfy the usual restrictions on net-revenue functions such as symmetry. Here we focus on the additional cross-revenue-function restrictions that must be imposed if the fictitious-payoff function is to exist. These restrictions, which are very simple, consist of

$$(18) \quad \delta^1_{ij} = \delta^2_{ji}, \qquad \text{for all } i \text{ and } j.$$

The net-revenue function (17), which is linear in parameters, is ideally suited to empirical implementation. Imposition of the restrictions (18) poses no problems from a practical point of view.

It is now time to use the fictitious-payoff function.

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8. The modifications required for more than two firms are straight forward.

3 The First Application: A Graphical Analysis of Strategic Behavior in Dynamic Games

In this section, the fictitious-payoff function is used in a graphical analysis of a dynamic game. Open and (subgame-perfect) closed-loop equilibria are calculated and the corresponding fictitious-objective functions are graphed. These are then compared to monopoly and surplus-maximizing (real) objective functions and outcomes.

The particular example studied is one of learning by doing, where each firm's cost falls with its cumulative production. The payoff functions for this game are given by equation 2 with x replaced by q. It is useful to begin with an analysis of the first-order condition for the maximization of $\Pi^i(q_1, q_2)$ with respect to q_1^i,

$$\begin{aligned}(19)\qquad \partial(\pi_1^i+\pi_2^i)/\partial q_1^i &= \partial\pi_1^i/\partial q_1^i + \partial\pi_2^i/\partial q_1^i + (\partial\pi_2^i/\partial q_2^i)(\partial q_2^i/\partial q_1^i)\\ &\quad + \sum_{j\neq i}(\partial\pi_2^i/\partial q_2^j)(\partial q_2^j/\partial q_1^i)=0.\end{aligned}$$

The first expression between the equal signs of (19) is the static effect. The Nash equilibrium for the one-shot game [the solution to the maximization (1)] is obtained by setting this expression equal to zero.

The second expression between the equal signs of (19) is the pure intertemporal effect. This effect, which is due to the cost reduction associated with higher q_1^i, is recognized when calculating both open and closed-loop equilibria.

The third expression equals zero in both open and closed-loop first-order conditions. With the open-loop calculation, $\partial q_2^i/\partial q_1^i$ is zero, whereas with the closed-loop calculation, $\partial\pi_2^i/\partial q_2^i$ is zero by the first-order conditions for the period-two maximization.

It is the final expression that distinguishes the two equilibrium concepts. The open-loop calculation, which is essentially static, ignores the fact that first-period choices alter second-period incentives. This effect, which I denote the dynamic-strategic effect, is recognized in the calculation of the closed-loop equilibrium. Dynamically strategic players anticipate the outcome of the continuation game in making their period-one choices.

3.1. Learning by Doing, Symmetric Duopoly

Consider a symmetric duopoly that produces a homogeneous product and plays a two-period game. It is assumed that a linear inverse-demand function determines an industry-wide price and that marginal cost is constant in each period. Under these circumstances, firm i's period-t profit is

$$\pi_t^i = (a - Q_t - c_t^i)\, q_t^i, \qquad Q_t := q_t^1 + q_t^2, \quad i = 1, 2, \quad t = 1, 2. \tag{20}$$

In period one, each firm's marginal cost is the same constant c_1. The two periods are linked through second-period marginal costs, which are

$$c_2^i = \alpha - \beta\, q_1^i, \qquad \alpha > 0, \quad \beta > 0. \tag{21}$$

(21) indicates that i's period-two cost falls as its period-one output increases.

We begin with the calculation of the closed-loop equilibrium, starting with period two. In this period, firm i chooses q_2^i to maximize π_2^i, conditional on q_2^j and both period-one outputs. First-order conditions for this maximization are

$$a - 2\, q_2^i - q_2^j - c_2^i = 0. \tag{22}$$

Solving the two first-order conditions, we obtain

$$q_2^i = (a - 2\, c_2^i + c_2^j)/3 \tag{23}$$

which, when substituted into (20) yields

$$\pi_2^{i*} = (a - 2\, c_2^i + c_2^j)^2/9 = (a - \alpha + 2\,\beta\, q_1^i - \beta\, q_1^j)^2/9. \tag{24}$$

Anticipating the outcome (24), in the first-period firm i chooses q_1^i to maximize $\pi_1^i(q_1) + \pi_2^{i*}(q_1)$. In the notation of proposition one,

$$\begin{aligned} \Psi^i(q_1) &= \pi_1^i(q_1) + \pi_2^{i*}(q_1) \\ &= \{-q_1^1 q_1^2 + (a - c_1 - q_1^i)\, q_1^i + 0\} \\ &\quad + \{-4\,\beta^2 q_1^1 q_1^2 + 4\,\beta(a - \alpha)\, q_1^i + 4\,\beta^2 (q_1^i)^2 \\ &\quad + (a - \alpha)^2 - 2\,\beta(\alpha - \alpha)\, q_1^j + \beta^2 (q_1^j)^2\}/9. \end{aligned} \tag{25}$$

In equation 25, the first and second-period profit functions, $\pi_1^i(q_1)$ and $\pi_2^{i*}(q_1)$, have been separated into their three parts, $\Psi(q_1)$, $\Omega^i(q_1^i)$, and $\Theta^i(q_1^{-i})$. We can therefore use proposition one to write the closed-loop fictitious-payoff function, F^{CL}. It is convenient to split F^{CL} into two parts, F_1^{CL} and F_2^{CL}. F_1^{CL} is associated with π_1^i whereas F_2^{CL} is associated with π_2^{i*}.

$$\begin{aligned} F^{CL} = F_1^{CL} + F_2^{CL} &= \{-q_1^1 q_1^2 + \sum_i (a - c_1 - q_1^i)\, q_1^i\} \\ &\quad + \{-4\,\beta^2 q_1^1 q_1^2 + \sum_i [4\,\beta(a - \alpha)\, q_1^i + 4\,\beta^2 (q_1^i)^2]\}/9. \end{aligned} \tag{26}$$

To obtain the symmetric solution, we set $q_1^1 = q_1^2 = \mathbf{q}_1$ in (26). The result is[9]

$$\begin{aligned} F^{CL} &= F_1^{CL} + F_2^{CL} \\ &= \{2\,(a - c_1)\,\mathbf{q}_1 - 3\,\mathbf{q}_1^2\} + \{8\,\beta(a - \alpha)\,\mathbf{q}_1 + 4\,\beta^2\,\mathbf{q}_1^2\}/9. \end{aligned} \tag{27}$$

9. With a slight abuse of notation, F^{CL} denotes both the two-variable payoff function (26) and the symmetric one-variable payoff function (27).

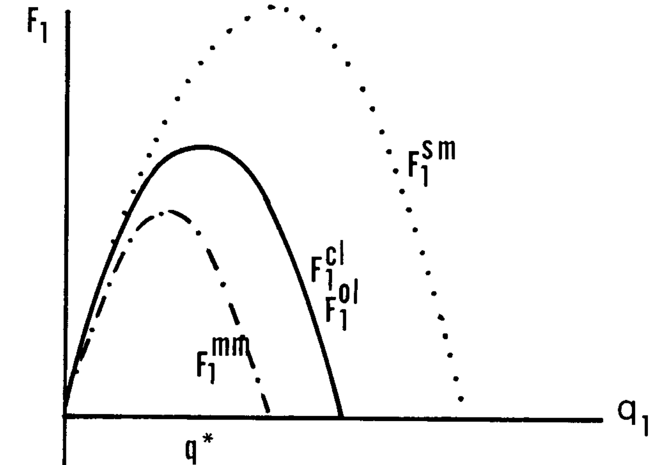

FIGURE 1

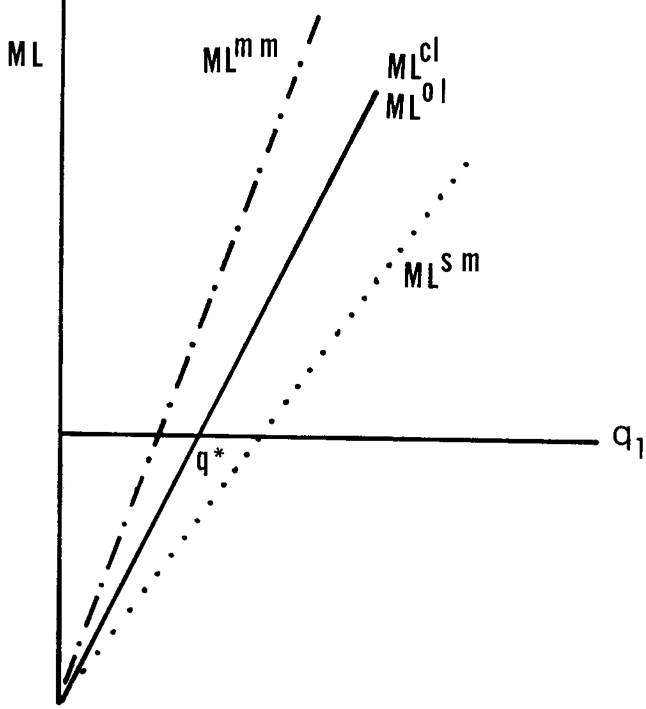

FIGURE 2

Period One.

We can now see what the market "maximizes". F_1^{CL} and F_2^{CL} are plotted with solid lines in figures 1 and 3 respectively.[10] F_1^{CL} initially increases with $\mathbf{q}_1$ but eventually falls. The marginal loss due to increasing output today, ML^{CL}, which is the negative of the slope of F_1^{CL}, is shown in figure 2. ML^{CL} equals zero at $\mathbf{q^*}$, the Nash equilibrium of the one-shot

10. The graphs correspond to the following parameter values: $a=9$, $\alpha=6$, $\beta=1$, and $c_1=3$.

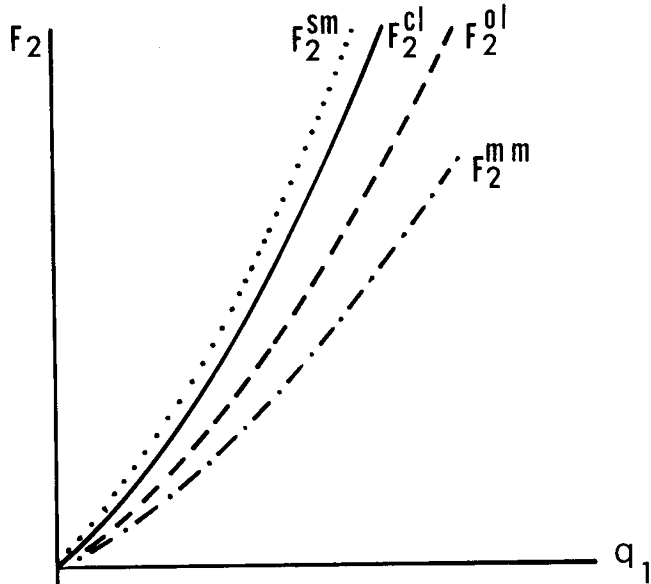

FIGURE 3

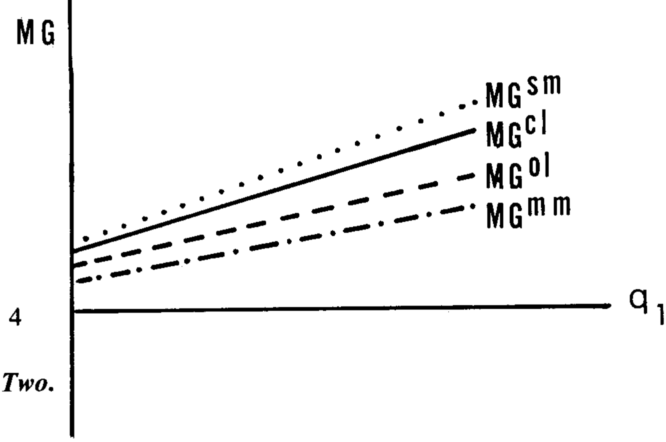

FIGURE 4

Period Two.

game. As figure 3 shows, F_2^{CL} increases monotonically with $\mathbf{q}_1$. The marginal future gain due to an increase in today's output, MG^{CL}, which is the slope of F_2^{CL}, is graphed in figure 4.

Finally, figure 5 unites the two periods. The closed-loop-equilibrium period-one output is determined by the intersection of ML^{CL} and MG^{CL}. This output is denoted $\mathbf{q}_1^{CL}$. The figure shows that $\mathbf{q}_1^{CL}$ is greater than $\mathbf{q}^*$. When there is a future, firms produce at higher rates in the first period in order to lower costs in the second.

Similar calculations can be performed to obtain the open-loop fictitious-payoff function, F^{OL}. Open-loop outputs are found by solving the maximization

$$\max_{q_1^i,\, q_2^i} \Psi^i(q_1, q_2) = (a - q_1^i - q_1^j - c_1)\, q_1^i + (a - q_2^i - q_2^j - \alpha + \beta q_1^i)\, q_2^i, \quad i = 1, 2, \quad j = 2, 1. \tag{28}$$

Once more we seek a symmetric solution. Calculations similar to those used to obtain F^{CL} yield

$$(29) \qquad F^{OL} = F_1^{OL} + F_2^{OL}$$
$$= \{2(a-c_1)\mathbf{q}_1 - 3\mathbf{q}_1^2\} + \{2\beta(a-\alpha)\mathbf{q}_1 + \beta^2\mathbf{q}_1^2\}/3.$$

Static fictitious-payoff functions F_1^{CL} and F_1^{OL} corresponding to the two solution concepts are the same, as is indicated in figures 1 and 2. However, as the dashed graphs in figures 3 and 4 show, the dynamic portions differ. Figure 4 indicates that when duopolists are dynamically strategic they value future cost reductions more. That is to say, MG^{CL} lies everywhere above the open-loop marginal gain, MG^{OL}.

The intuition behind this result is as follows. Firm i increases its first-period output in order to lower its second-period cost. As a result, i's second-period output increases. Moreover, this increase in i's period-two output causes j's period-two output to fall, further increasing i's profit.[11] This is the dynamic-strategic factor. Oligopolistic rivalry in the second period reinforces the pure intertemporal effect and causes still higher output in period one.

Figure 5 shows that the open-loop period-one output $\mathbf{q}_1^{OL}$, which is determined by the intersection of ML^{OL} and MG^{OL}, is less than its closed-loop counterpart, $\mathbf{q}_1^{CL}$. This is a direct result of the lower valuation of $\mathbf{q}_1$ in the open-loop equilibrium.

Suppose now that a monopolist controls the two plants that were assumed to be owned by separate firms in the previous calculations. The monopolist has a real (not a fictitious) objective function, which can also be located on the graphs. Denote this function F^{MM}, where MM stands for multiplant monopolist.

The monopolist's problem is to

$$(30) \qquad \max_{Q_1,\, Q_2} \; (a-Q_1-c_1)Q_1 + (a-Q_2-c_2)Q_2,$$

where

$$(31) \qquad c_2 = \alpha - \beta q_1^i = \alpha - \beta q_1^j = \alpha - \beta Q_1/2.$$

In writing (31) is was assumed that the monopolist splits his total output equally between his two plants and that learning is plant specific.

Again a symmetric solution is sought. In terms of $\mathbf{q}_1$, the monopolist's objective function is

$$(32) \qquad F^{MM} = F_1^{MM} + F_2^{MM}$$
$$= \{2(a-c_1)\mathbf{q}_1 - 4\mathbf{q}_1^2\} + \{2\beta(a-\alpha)\mathbf{q}_1 + \beta^2\mathbf{q}_1^2\}/4.$$

11. In the language of BULOW, GEANAKOPLOS, and KLEMPERER [1985], choice variables here are strategic substitutes.

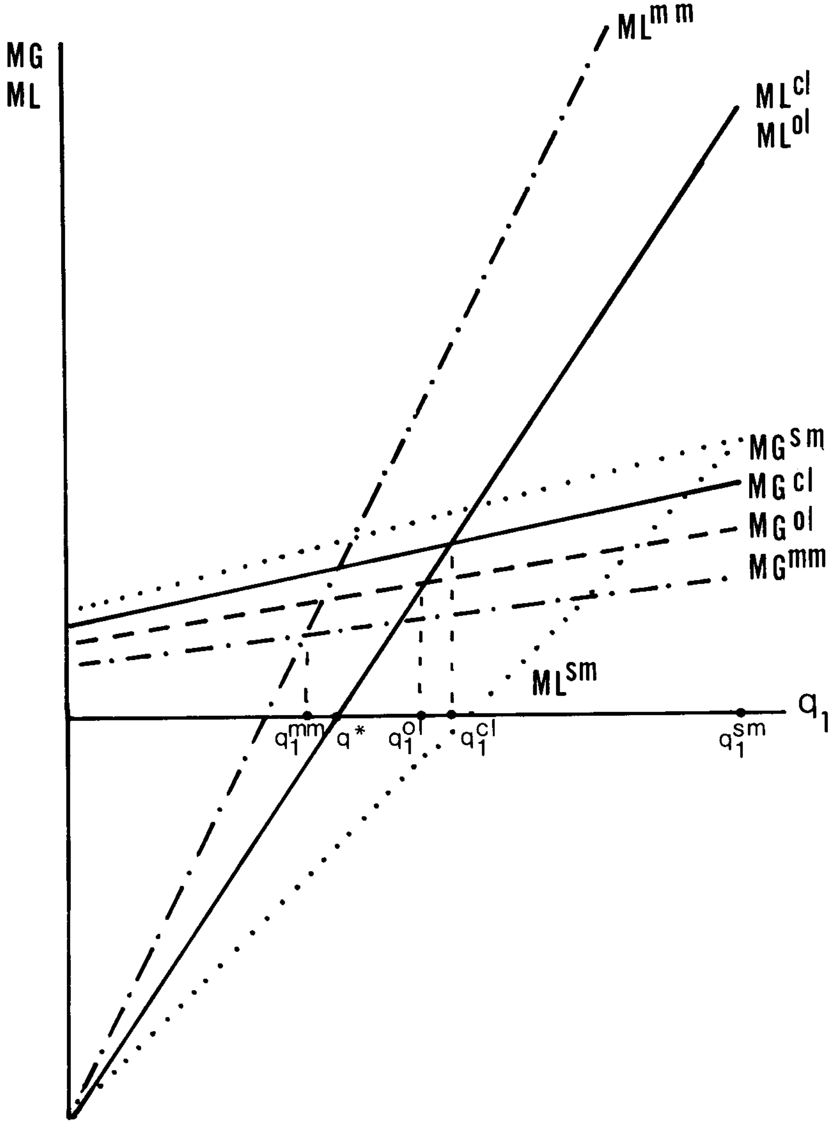

FIGURE 5

Solutions to the Game.

F_1^{MM} is the graph with alternating dots and dashes in figure 1. As is expected, this function obtains its maximum at a lower value of $\mathbf{q}_1$ than $\mathbf{q}^*$, the Nash solution to the one-shot game. The monopolist's marginal-loss function, the negative of the slope of F_1^{MM}, lies everywhere above the duopolists' ML function. This can be seen in figure 2. Clearly, in the static game the monopolist places a lower marginal value on each unit of output.

Turning to figure 3, we see than F_2^{MM}, lies below both F_2^{OL} and F_2^{CL}. Similarly, MG^{MM} lies below MG^{OL} and MG^{CL}, as can be seen in figure 4. Finally, figure 5 shows that period-one output is lower when the industry is controlled by a monopolist than with either duopoly-solution concept. These results are as expected.

The final solution concept involves a planner or surplus maximizer. Like the monopolist, the surplus maximizer has a real (not a fictitious) objective function. She seeks to maximize the sum of profits and consumer surplus. We denote this function F^{SM}, where SM stands for surplus maximizer.

Given the linearity and unitary slope of our demand function, period-t consumer surplus is $1/2\,Q_t^2$. The planner's problem is therefore to

$$\max_{Q_1,\,Q_2}\ (a-Q_1-c_1)Q_1+1/2\,Q_1^2 + (a-Q_2-\alpha+\beta Q_1/2)Q_2+1/2\,Q_2^2. \tag{33}$$

Just as with the monopolist, it is assumed that the surplus maximizer splits total production equally between her two plants and that learning is plant specific.

In terms of $\mathbf{q}_1$, the planner's objective function is

$$\begin{aligned} F^{SM} &= F_1^{SM}+F_2^{SM} \\ &= \{2(a-c_1)\mathbf{q}_1-2\mathbf{q}_1^2\}+\{2\beta(a-\alpha)\mathbf{q}_1+\beta^2\mathbf{q}_1^2\}/2. \end{aligned} \tag{34}$$

F_1^{SM} is the dotted graph in figure 1. As is expected, it obtains a maximum at a higher level of $\mathbf{q}_1$ than do the other static-objective functions. And, as can be seen in figure 2, the static marginal loss due to increasing $\mathbf{q}_1$, ML^{SM} is lower for the surplus maximizer. The higher valuation of $\mathbf{q}_1$ in the static problem results from the fact that the planner is sensitive to the surplus that consumers receive from the additional output.

Turning to figure 3, it can be seen that the surplus maximizer also places a higher dynamic value on $\mathbf{q}_1$. That is to say, F_2^{SM}, lies everywhere above the other F_2's. As before, this results from the fact that the planner values the surplus that consumers receive from $\mathbf{q}_2$ and therefore plans to produce at higher levels in the second period.

Finally, figure 5 shows the determination of equilibrium $\mathbf{q}_1$ for the surplus maximizer. Given that $\mathbf{q}_1$ is valued more for both static and dynamic reasons, it is not surprising that $\mathbf{q}_1^{SM}$ is considerably greater than either duopoly or monopoly first-period output.

3.2. The Effect of N

The analysis of the last subsection showed that when there are only two players, the dynamic portion of the fictitious-payoff function can differ considerably depending on duopoly-solution concept. In particular, the cost-reducing feature of period-one output is valued more in the closed-loop equilibrium than in the open-loop equilibrium. It is of interest to determine whether this difference disappears as the number of players grows. The question is, do F_2^{CL} and F_2^{OL} start to coincide as N increases?

To answer this question, we first consider the N-player closed-loop equilibrium. Firm i's period-t profit is now

$$\pi_t^i = (a - Q_t - c_t^i)\, q_t^i, \qquad Q_t := \sum_j q_t^j, \tag{20'}$$
$$i = 1, \ldots, N, \qquad j = 1, \ldots, N, \qquad t = 1, 2.$$

In the second period, firm i chooses q_2^i to maximize π_2^i. The equilibrium period-two output conditional on period-one choices is

$$q_2^i = (a - N c_2^i + \sum_{j \neq i} c_2^j)/(N+1) \tag{23'}$$

and corresponding profit is

$$\pi_2^{i*} = (a - N c_2^i + \sum_{j \neq i} c_2^j)^2/(N+1)^2 \tag{24'}$$
$$= (a - \alpha + N \beta q_1^i - \beta \sum_{j \neq i} q_1^j)^2/(N+1)^2.$$

In the first period, firm i chooses q_1^i to maximize $\pi_1^i + \pi_2^{i*}$. Calculations similar to those performed before yield the symmetric N-player closed-loop fictitious-payoff function, $F^{CL}(N)$,

$$F^{CL}(N) = F_1^{CL}(N) + F_2^{CL}(N) = \{N(a - c_1)\mathbf{q}_1 - N(N+1)\mathbf{q}_1^2/2\} \tag{27'}$$
$$+ \{2N^2 \beta (a - \alpha)\mathbf{q}_1 + N^2 \beta^2 \mathbf{q}_1^2\}/(N+1)^2.$$

The N-player open-loop maximization yields the N-player open-loop fictitious-payoff function, $F^{OL}(N)$,

$$F^{OL}(N) = F_1^{OL}(N) + F_2^{OL}(N) = \{N(a - c_1)\mathbf{q}_1 - N(N+1)\mathbf{q}_1^2/2\} \tag{29'}$$
$$+ \{N \beta (a - \alpha)\mathbf{q}_1 + N \beta^2 \mathbf{q}_1^2/2\}/(N+1).$$

Just as before, the static portions are the same but the dynamic portions differ across solution concepts. F_2^{CL} and F_2^{OL} are reproduced in figure 6 for $N=2$. To determine the effect of N, $F_2^{OL}(N)$ is held constant as N increases. This is accomplished by multiplying both $F_2^{CL}(N)$ and $F_2^{OL}(N)$ by $(N+1)/2N$, which clearly preserves their relative importance. In figure 6, therefore, only $F_2^{CL}(N)$ is shown for N greater than 2.

Figure 6 shows that as N increases, the two fictitious-payoff functions diverge by larger and larger amounts. The intuition for this result is as follows. In calculating the open-loop equilibrium, firms behave as if future as well as present rival output will be unaffected by today's choice. As the number of rivals increases, failure to anticipate one's own effect on future rival behavior becomes more and more important. This can be seen in equation 19. The dynamic-strategic effect is essentially multiplied by the number of rivals whereas the intertemporal effect is not.

The fact that the objective functions diverge does not mean that the equilibrium output vectors diverge. The game described here satisfies the conditions under which open and closed-loop equilibria are approximately equal for large N (Fudenberg and Levine [1988]). In the limit, therefore, $\mathbf{q}_t^{OL}$ and $\mathbf{q}_t^{CL}$ must coincide. In fact, they both approach zero.

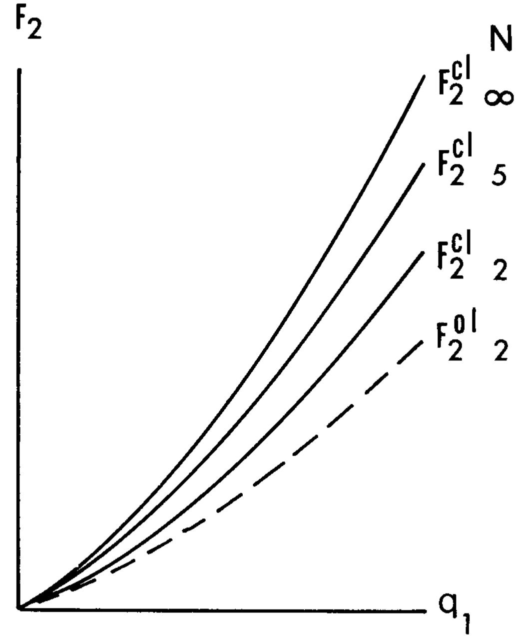

FIGURE 6

The Effect of N in Period Two.

4 The Second Application: Calculation of Nash Equilibria

The second application is to the calculation of Nash equilibria of large games. When the restrictions of proposition one are satisfied so that a fictitious-payoff function exists, it easy to calculate Nash equilibria, both static and subgame perfect. [12] This calculation is reduced to an ordinary optimization problem and standard mathematical-programming techniques can be used. In this section, the fictitious-payoff method of computing Nash equilibria is compared to other algorithms and two simple examples are presented.

The fact that maximizing the fictitious-payoff function and calculating Nash equilibria of the game are closely related is fairly obvious. Nevertheless, I state this relationship formally as

12. To calculate subgame-perfect equilibria using the fictitious-payoff function, one must solve the second-stage optimization first. In some games, this is not an easy problem.

PROPOSITION 2: When condition (10) is satisfied so that a fictitious-payoff function exists,[13]

(*a*) necessary conditions for a maximum of F and for a Nash equilibrium of the game are

$$(35) \qquad \partial\Psi^i(x)/\partial x^i = \partial \mathrm{F}(x)/\partial x^i = 0, \qquad i = 1, \ldots, n,$$

(*b*) sufficient conditions for a maximum of F are that the determinants of the principal minors of the Hessian of F alternate in sign, where the sign of the first principal minor is negative,

(*c*) sufficient conditions for a pure-strategy-Nash equilibrium of the game are that the diagonal elements of the Hessian of F be negative.

Discussion:

(*a*) is true by construction.

(*b*) is the standard condition for a maximum of a twice-differentiable function.

(*c*) is a weaker condition than (*b*). For a Nash equilibrium, it is sufficient that $\partial^2 \pi^i/(\partial x^i)^2 = \mathrm{F}_{ii} < 0$ for all i (diagonal concavity). A maximum of F, in contrast, requires full concavity.

The implication of proposition 2 is that all pure-strategy Nash equilibria can be found by finding the stationary points of F. In addition, all local maxima of the fictitious-payoff function are equilibria of the game (but not vice versa). Finally, the second-order conditions for a Nash equilibrium are easier to check than the second-order conditions for a maximum of F.

One might, however, ask why this is an aid in calculating equilibria. It is also possible to obtain the Nash equilibria by solving the system of equations (8) for fixed points. I believe that there are two advantages to the optimization approach. The first involves the amount of information that can be used in the calculation. Typically, optimization programs use the objective-function-gradient vector to determine the direction in which to move and the value of the objective function to determine the step length. As a consequence, convergence properties of optimization programs are usually superior to those of programs that solve systems of equations.

The second advantage concerns the software that is routinely available. Almost all optimization programs incorporate sophisticated methods of dealing with linear and nonlinear constraints. Unfortunately, this is often not the case with programs that solve systems of equations.

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13. The proposition concerns interior equilibria. It remains true, however, for constrained equilibria (see example one in section 4.2.).

4.1. Comparison with Other Algorithms

A number of algorithms have been proposed for computing Nash equilibria with twice-differentiable payoff functions. Three are discussed here – those due to ROSEN [1965], MURPHY, SHERALI, and SOYSTER [1982], and KOLSTAD and MATHIESEN [1988]. All of these methods can be used to calculate equilibria of games that do not satisfy condition (10) and are thus more general. The object of the comparison is to emphasize the computational savings that can be achieved when the restrictions are imposed.

ROSEN [1965] points out that one can use the n first-order conditions (8) as if they were the gradient of a function to be maximized, even when this function does not exist. In this "optimization" problem, the fictitious gradient gives the direction in which to move. The problem lies in choosing the step size. Many mathematical-programming algorithms choose the step length so as to give a maximum value of the true function along the chosen ray. The fictitious-gradient method, however, chooses the step length so as to minimize the norm of the projection of the fictitious gradient onto the constraint set.

To use Rosen's method, it is therefore necessary to modify the internal calculations performed by the mathematical-programming software. To use the fictitious-payoff method, in contrast, no such modifications are required. Any standard nonlinear-optimization subroutine can be used as it is. In addition, we know much more about convergence properties of standard optimization algorithms than about convergence to Nash equilibria.

Both MURPHY, SHERALI, and SOYSTER [1982] and KOLSTAD and MATHIESEN [1988] propose iterative techniques for computing Cournot-Nash equilibria. Unlike Rosen's method, which is a single programming problem, their techniques involve solving a family of convex programs. Murphy, Sherali, and Soyster's algorithm iterates between linear approximations to the demand curve and a mathematical program containing the demand-curve approximation and the supply structure. KOLSTAD and MATHIESEN's [1988] method, in contrast, involves linearizing the first-order conditions (8) at each iteration and solving the resulting linear-complementarity problem using LEMKE's [1965] algorithm. Unlike fictitious payoff and fictitious gradient, these methods are limited to calculating Cournot-Nash equilibria of games with additively separable revenues and costs.

Finally, one can simply solve the system of nonlinear equations (8) directly. This approach has been successful in solving large-scale applied-general-equilibrium models (see for example SCARF [1984]). Many of these models, however, are based on the assumption of constant returns to scale and make use of the special structure of competitive equilibria (such as price equals average cost).

Given the simplicity of the fictitious-payoff calculations and the fact that more complex solution concepts and revenue/cost structures can be handled, the imposition of restrictions (10) may be a small price to pay for the efficiency gained. This is particularly true for very large models.

4.2. Some Examples

Nash equilibria for two games were calculated using the University of British Columbia's NLP nonlinear optimization software (1984). With both examples, convergence to a Nash equilibrium was achieved in less than 1/10 of a second at a cost of approximately 5¢ per problem.

The first game is an N-firm Cournot-Nash problem with asymmetric quadratic costs. There are five players, each with profit function

$$\pi^i = (a - \sum_j q^j)\, q^i - \alpha^i q^i - \beta^i (q^i)^2, \qquad i = 1, \ldots, 5. \tag{36}$$

The corresponding fictitious-payoff function is

$$F = \sum_i [(a - q^i)\, q^i - \alpha^i q^i - \beta^i (q^i)^2 - 1/2 \sum_{j \neq i} q^i q^j]. \tag{37}$$

Table 1 shows parameter values and the corresponding Nash equilibria of the game. Notice that in equilibrium, firm four does not produce. Imposition of constraints such as $q^i \geqslant 0$, $i = 1, \ldots, N$, is entirely straight forward.

The second example is an asymmetric-cost learning-by-doing game. The model of subsection 3.1. is modified so that (constant) period-one marginal costs are c_1^i, $i = 1, 2$, and period-two marginal costs are

$$c_2^i = \alpha^i \exp(-\beta^i q_1^i), \qquad \alpha^i > 0, \quad \beta^i > 0. \tag{21''}$$

In the closed-loop equilibrium, period-two profit is

$$\pi_2^{i*} = [a - 2\,\alpha^i \exp(-\beta^i q_1^i) + \alpha^j \exp(-\beta^j q_1^j)]^2/9. \tag{24''}$$

In period one, firm i chooses q_1^i to maximize

$$\max_{q_1^i}\, (a - q_1^i - q_1^j - c_1^i)\, q_1^i + [a - 2\,\alpha^i \exp(-\beta^i q_1^i) + \alpha^j \exp(-\beta^j q_1^j)]^2/9. \tag{38}$$

This process yields

$$\begin{aligned} F^{CL} = & -q_1^1 q_1^2 + \sum_i [(a - c_1^i\, q_1^i)\, q_1^i] \\ & + \{-4\,\alpha^1 \exp(-\beta^1 q_1^1)\, \alpha^2 \exp(-\beta^2 q_1^2) \\ & + \sum_i [(a - 2\,\alpha^i \exp(-\beta^i q_1^i))^2]\}/9. \end{aligned} \tag{26''}$$

Table 2 shows parameter values and corresponding closed-loop period-one and period-two outputs. Again we see that the low-cost firm (firm one) produces more in the first period than its high-cost rival. As a result, period-two production is even more asymmetric than production in period-one.

5 Final Remarks

This paper presents the fictitious-payoff function and provides necessary and sufficient conditions for its existence. Clearly, not all games satisfy these conditions. But for games that do, the fictitious-payoff function is a useful tool. Its main methodological advantage stems from the fact that methods for solving optimization exercises are substantially more advanced than methods for solving equilibrium problems.

Two applications illustrate the use of the fictitious-payoff function. The first is a graphical abalysis of strategic behavior in a dynamic game. Here, fictitious open and closed-loop objective functions are derived and graphed. These solution concepts are then compared to multiplant-monopolist and surplus-maximizing objective functions. The latter two functions, unlike the fictitious-payoff function, are familiar economic tools.

The second application is to numerical calculation of Nash equilibria of large models with imperfect competition. Here, the two examples involve at most five players. The general method, however, can be used to calculate asymmetric subgame-perfect equilibria of very large games.

The two applications discussed are by no means exhaustive; there are many other potential and actual uses for the fictitious-payoff function. For example, the general formulation of a flexible-functional form for net-revenue functions which is given in equation 17 is entirely suitable for econometric estimation. In addition to the usual cross-equation restrictions that must be satisfied by any net-revenue function, the cross-revenue-function restrictions (18) must be imposed if an oligopolistic-market-objective function is to exist.

The fictitious-payoff function is also used in SLADE [1988 *b*] to illustrate why dynamic oligopolists facing capacity constraints may choose to ration customers in early periods rather than serve the entire market. This game is similar to the learning-by-doing example of the current paper. The intertemporal link, however, is through each firm's demand intercept rather than through its marginal-cost function. The use of a single objective function rather than N firm-specific-payoff functions simplifies the computations considerably.

Finally, the fictitious-payoff function is a useful tool for teaching as well as for research. The ability to show what the market "maximizes" in a wide class of games can help students to understand the nature of strategic interaction and the inefficiencies that such behavior introduces.

APPENDIX

Proof of Proposition One

PROPOSITION 1: The fictitious-payoff function F (x) exists if and only if the firm payoff functions $\Psi^i(x)$ can be written as

$$(A_1) \qquad \Psi^i(x) = \Psi(x) + \Omega^i(x^i) + \Theta^i(x^{-i}), \qquad i = 1, \ldots, n.$$

When (A_1) is satisfied, the fictitious-payoff function is

$$(A_2) \qquad F(x) = \Psi(x) + \sum_i \Omega^i(x^i)$$

Proof: First, assume that (A_1) is true. First-order conditions for the maximization of Ψ^i are then

$$(A_3) \qquad \Psi^i_i = \Psi_i(x) + \Omega^i_i(x^i) = 0.$$

By definition, F exists if the n first-order conditions (A_3) are integrable. This will be true if $\Psi^i_{ij} = \Psi^j_{ji} = \Psi_{ij}$. Partially differentiating (A_3) with respect to x^j, we see that

$$(A_4) \qquad \Psi^i_{ij} = \Psi_{ij}(x),$$

which clearly equals Ψ^j_{ji}. F therefore exists.

Now assume that F exists. By the definition of F, its existence implies that the first-order conditions (8) are integrable. This in turn implies that $\Psi^i_{ij} = \Psi^j_{ji} = \Psi_{ij}$, for all i and j. Integrating Ψ^i_{ij} with respect to x^j yields

$$(A_5) \qquad \Psi^i_i = \Psi_i(x) + \Phi^i_i(x^{-j}).$$

Integrating (A_5) with respect to x^i we have

$$(A_6) \qquad \Psi^i = \Psi(x) + \Phi^i(x^{-j}) + \Theta^i(x^{-i}).$$

Partially differentiating (A_6) with respect to x^k we obtain

$$(A_1) \qquad \Psi^i_k = \Psi_k(x) + \Phi^i_k(x^{-j}) + \Theta^i_k(x^{-i}),$$

and partially differentiating (A_1) with respect to x^i yields

$$(A_8) \qquad \Psi^i_{ik} = \Psi_{ik}(x) + \Phi^i_{ik}(x^{-j}).$$

Now consider firm k. The analogues to (A_5)-(A_8) are

$$(A_9) \qquad \Psi^k_k = \Psi_k(x) + \Phi^k_k(x^{-j}),$$

$$(A_{10}) \qquad \Psi^k = \Psi(x) + \Phi^k(x^{-j}) + \Theta^k(x^{-k}),$$

$$(A_{11}) \qquad \Psi^k_i = \Psi_i(x) + \Phi^k_i(x^{-j}) + \Theta^k_i(x^{-k}),$$

$$(A_{12}) \qquad \Psi^k_{ik} = \Psi_{ik}(x) + \Phi^k_{ik}(x^{-j}).$$

(A_{12}) will equal (A_8) if and only if

(A_{13}) $$\Phi^i_{ik}(x^{-j}) = \Phi^k_{ik}(x^{-j}) = \Phi_{ik}(x^{-j})$$

or

(A_{14}) $$\Phi^i_{ik}(x^{-j}) = \Phi^k_{ik}(x^{-j}) = 0.$$

When (A_{13}) is true, Ψ^i contains no firm-specific function of x^{-j}, and when (A_{14}) is true, x^k does not enter Φ^i. k, however, was an arbitrary firm different from i and j. If Ψ^i contains firm-specific functions, therefore, they are of the form

(A_{15}) $$\Phi^i(x^{-j}) = \Omega^i(x^i) \quad \text{and} \quad \Theta^i(x^{-i}).$$

The fictitious-payoff function F is not unique. [14] To show that (A_2) is a candidate for F, it suffices to show than its gradient vector satisfies equation (9). Using (A_2) we have

(A_{16}) $$\partial F(x)/\partial x^i = \Psi_i(x) + \Omega^i_i(x) = \partial \Psi^i(x)/\partial x^i,$$

where the second equality in (A_{16}) results from equation (A_3). □

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