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On the Eigenvectors of Macro-Economic Models

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ABSTRACT. – In this study we examine the eigenvectors pertaining to linear economic models. In particular, we propose a procedure to investigate whether it is possible to describe the vector consisting of the endogenous variables of a given model well in terms of a small number of eigenvectors. Next, we discuss an approach to analyze the sensitivity of those eigenvectors with respect to the model coefficients. Our procedures are illustrated by an application to an empirical model.

Sur les vecteurs propres des modèles macroéconomiques

RÉSUMÉ. – Dans cette étude nous examinons les vecteurs propres d'un modèle macroéconomique. En particulier, nous proposons une procédure pour étudier s'il est possible de bien décrire le vecteur des variables endogènes d'un modèle donné à l'aide d'un petit nombre de vecteurs propres. Ensuite nous discutons une approche permettant d'analyser la sensibilité du modèle. Nos procédures sont illustrées par une application à un modèle empirique.

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1 Introduction

It is well-known that the dynamic behaviour of a linear macro-economic model depends on the solution of the eigenvalue problem of a matrix associated with the reduced form of such a model. If the relevant matrix is diagonable, then the vector consisting of the endogenous variables, i. e. the variables to be explained by the model, can be interpreted as being a linear combination of the complete set of eigenvectors. In this paper, we shall focus attention on the possibility to represent the vector of endogenous variables adequately by a small subset of these eigenvectors. Next, we investigate the sensitivity of those vectors with respect to changes in the model coefficients. We propose a helpful procedure which is based on the inspection of the effect of changes in all coefficients associated with a lagged endogenous variable. This approach extends in an uniform way to the study of eigenvectors, the similar procedures used to investigate the sensitivity of the eigenvalues as proposed by DELEAU and MALGRANGE [1975, 1976], SCHOONBEEK [1984] and the references given there. The above methods form a tool to get insight into the time paths followed by the endogenous variables and help to identify what really matters in a model.

As a starting point we take the first-order dynamic linear economic model (or the deterministic part of an estimated econometric model of that type). See for the treatment of linear economic models GANDOLFO [1980]. As a consequence of the linearity of the models, results of the theory of matrices can be used in this context. In particular ANDRIGHETTO, OUDET and GUÉRIN [1975], and recently very extensively, KUH and NEESE [1982] and KUH, NEESE and HOLLINGER [1985] discuss the relevance of the spectral decomposition of a matrix, in relation to the above mentioned eigenvalue problem. Using this decomposition the dynamic responses of a model are written as a sum of terms, each of which is related to one specific eigenvalue-eigenvector combination. By inspection of the relative magnitudes of these terms, Kuh *et al.* identify the dominant ones. Next they investigate the eigenvalues associated with the latter terms in detail. We shall further analyze the just mentioned spectral decomposition. However, as said above, we will concentrate on the role of the eigenvectors. We discuss in section 2 a method to locate the eigenvectors which contribute relatively much to the values of the endogenous variables. In section 3, we discuss the sensitivity analysis of the eigenvectors. An application of our procedures to an empirical (Keynesian) economic model is described in section 4.

2 The Model and the Eigenvectors

We consider the following dynamic linear model

(1) $$\mathrm{A}\, y_t = \mathrm{B}\, y_{t-1} + \mathrm{C}\, x_t, \qquad t = 1, 2, \ldots$$

where y_t is the $(n \times 1)$ vector of the endogenous variables, and x_t the $(p \times 1)$ vector of the exogenous variables. The matrices $\mathrm{A} = (a_{ij})$, $\mathrm{B} = (b_{ij})$ and $\mathrm{C} = (c_{ij})$ have appropriate formats. We assume that the matrix A is regular. Then the reduced form of (1) reads

(2) $$y_t = \mathrm{D}\, y_{t-1} + \mathrm{E}\, x_t,$$

where $\mathrm{A}^{-1}\mathrm{B} \equiv \mathrm{D} = (d_{ij})$ and $\mathrm{A}^{-1}\mathrm{C} \equiv \mathrm{E} = (e_{ij})$. The eigenvalues of the matrix D will be denoted by λ_g $(g = 1, \ldots, n)$. In the following we assume that the model is dynamically stable, i. e. all λ_g are smaller than unity in absolute value. The right and left eigenvector of matrix D pertaining to λ_g are denoted as $r_g \equiv (r_{gh}) = (r_{g1}, \ldots, r_{gn})'$ and $l_g \equiv (l_{gh}) = (l_{g1}, \ldots, l_{gn})'$ respectively $(g = 1, \ldots, n)$. We further assume that the matrix D is diagonable, i. e. the n right eigenvectors (as well as the n left eigenvectors) are linearly independent. They are normalized such that $r_g' r_g = 1$ and $l_g' r_g = 1$ $(g = 1, \ldots, n)$.

From (2) we obtain after repeated substitutions

(3) $$y_t = \mathrm{D}^t y_0 + \sum_{s=0}^{t-1} \mathrm{D}^s \mathrm{E}\, x_{t-s}, \qquad t = 1, 2, \ldots$$

in which y_t is expressed in terms of the exogenous variables and starting position y_0. By using the spectral decomposition, see Ben-Israel and Greville [1974], of the matrices $\mathrm{D}^s (s = 0, 1, \ldots)$, we rewrite (3) as

(4) $$y_t = \sum_{g=1}^{n} r_g \left[\lambda_g^t l_g' y_0 + \sum_{s=0}^{t-1} \lambda_g^s l_g' \mathrm{E}\, x_{t-s} \right], \qquad t = 1, 2, \ldots$$

Thus, each y_t is expressed as a linear combination of the eigenvectors $r_g (g = 1, \ldots, n)$. The multiplicative factors, given between the square brackets in (4), clearly depend on the values of $y_0, x_1, \ldots, x_t$. Denote the factor associated with r_g in period t as $\alpha(g, t)$. Then, it is easy to see that $\alpha(g, t+1)$ is the sum of $\lambda_g . \alpha(g, t)$ plus a term which depends only on x_{t+1}. We remark further that an eigenvector pertaining to an eigenvalue which is identical to zero, also appears in a non-trivial way in (4). This becomes clear by considering the case $s = 0$ in the second summation in (4).

Now, we want to identify the eigenvectors r_g which contribute, in a sense to be specified yet, relatively much to the values of the endogenous variables. To begin with, consider the eigenvectors r_g where $g \in \mathrm{G}$, and G is an arbitrary real subset of the integers 1 up to n. We use this set to define the vectors y_t^G as

$$(5) \qquad y_t^G = \sum_{g \in G} r_g \left[\lambda_g^t l_g' y_0 + \sum_{s=0}^{t-1} \lambda_g^s l_g' E x_{t-s} \right], \qquad t = 1, 2, \ldots, T,$$

where T is a chosen value of t. Writing $y(T) \equiv (y_1', \ldots, y_T')'$, $y^G(T) \equiv (y_1^{G'}, \ldots, y_T^{G'})$, $x(T) \equiv (y_0', x_1', \ldots, x_T')$, we can summarize (4) and (5) as $y(T) = H(T)\, x(T)$ and $y^G(T) = H^G(T)\, x(T)$ respectively, where the matrices $H(T)$ and $H^G(T)$ are defined in an obvious way. A measure for the overall similarity of the vectors $y(T)$ and $y^G(T)$ is $\sup_{x(T) \neq 0} \| y(T) - y^G(T) \| / \| x(T) \|$, where $\| . \|$ is the quadratic vector norm. It is known that

$$(6) \qquad \sup_{x(T) \neq 0} \frac{\| H(T)\, x(T) - H^G(T)\, x(T) \|}{\| x(T) \|} = \| H(T) - H^G(T) \|_2$$

where the matrix norm $\| . \|_2$ denotes the spectral norm. See BEN-ISRAEL and GREVILLE [1974]. Thus, if the matrix $H(T)$ is approximated well by the matrix $H^G(T)$, then also the vectors $y(T)$ and $y^G(T)$ are close to each other, and vice versa. We note however, that in practice the computation of the above measure becomes very laborious and delicate if n, p and T are not very small. For instance, take the not unrealistic (moderate) values $n = 20$, $p = 10$ and $T = 8$. Then the corresponding matrices $H(8)$ and $H^G(8)$ are of dimension (160×90). Therefore, we propose to simplify the analysis by focusing the attention to the following extreme cases: (1) the "impact" case, where $T = 1$, and (2) the "limiting" case, where the exogenous variables remain constant, say $x_t = x$ for all t, and T goes to infinity. In the latter case, the vector of the endogenous variables converges to the vector y as given by

$$(7) \qquad y = F x,$$

where $F \equiv (I - D)^{-1} E$. Using the spectral decomposition of the matrix D, y is written as a combination of the vectors $r_g (g = 1, \ldots n)$

$$(8) \qquad y = \sum_{g=1}^{n} r_g \left(\frac{1}{1 - \lambda_g} l_g' E \right) x.$$

Analogous to the above discussion, we use the set G to define the vector y^G, and the matrix F^G as

$$(9) \qquad y^G = \sum_{g \in G} r_g \left(\frac{1}{1 - \lambda_g} l_g' E \right) x = F^G x.$$

The simplification we make implies that in order to evaluate the contribution of the eigenvectors r_g $(g \in G)$ to values of the endogenous variables, we inspect the magnitude of $\| F - F^G \|_2$.

Most interesting are the situations in which the set G contains only a few elements and where in the case (1) the vectors $y(1)$ and $y^G(1)$, in the case (2) the vectors y and y^G, are very similar. In those cases it is worthwhile to investigate the corresponding right eigenvectors in more detail.

Finally, we note that an eigenvector r_g can be a complex vector; i. e. $r_g = \mathrm{re}(r_g) + \mathrm{im}(r_g)\, i$. This complicates an interpretation in economic terms. However, we can always consider the 2 terms related to each other's complex conjugate eigenvectors together. It is easily demonstrated that then we have a linear combination of the (real) vectors $\mathrm{re}(r_g)$ and $\mathrm{im}(r_g)$.

3 The Sensitivity Analysis of the Eigenvectors

Suppose now that the eigenvector r_g has attracted our attention by the methods of section 2. In order to locate model coefficients which have a relatively large impact on this vector, we propose to analyze the sensitivity of the elements of this vector to changes in the model coefficients. Consider therefore the total differential

(10) $$dr_{gh} = \sum_i \sum_j \frac{\delta r_{gh}}{\delta d_{ij}} d_{ij}.$$

We refer to $w(r_{gh}, d_{ij}) \equiv \frac{\delta r_{gh}}{\delta d_{ij}} d_{ij}$ as the relative contribution of the coefficient d_{ij} to dr_{gh}. It can be interpreted as the change in r_{gh} as a result of a small relative change in d_{ij}. We propose to inspect in detail the relative magnitudes of the sums $\sum_i w(r_{gh}, d_{ij})$ for different j. We remark that this approach is similar to the procedures followed by DELEAU and MALGRANGE [1975, 1976] and SCHOONBEEK [1984] in order to investigate the sensitivity of the eigenvalues of the matrix D. The mentioned sums measure the effect of a same percentage change in all coefficients in the matrix D associated with the j-th lagged endogenous variable. It is easily verified that these sums also represent (apart from an opposite sign) the first-order effect of r_{gh} as a result of the replacement of the j-th column of matrix D by a column consisting of elements equal to zero. Thus, if we inspect the relative magnitudes of these sums, then, in fact, we inspect the first-order effect of the removal of the corresponding variables.

We can define analogously to $w(r_{gh}, d_{ij})$ the relative contributions $w(r_{gh}, a_{ij})$ and $w(r_{gh}, b_{ij})$. In the following lemma we present some identities for sums of relative contributions.

LEMMA : Let the matrix $D = A^{-1} B$ be diagonable and λ_g be a simple eigenvalue of this matrix with corresponding right eigenvector r_g. Consider the element r_{gh} of this vector. Then we have:

(i) $$\sum_i \sum_j w(r_{gh}, d_{ij}) = 0,$$

(ii) $\sum_i w(r_{gh}, d_{ij}) = \sum_i w(r_{gh}, b_{ij})$ if $j=1, \ldots, n$,

(iii) $\sum_j w(r_{gh}, d_{i*j}) = -\sum_i w(r_{gh}, a_{ij*})$, if $i* = j*$,

(iv) $\sum_j w(r_{gh}, b_{ij}) = -\sum_j w(r_{gh}, a_{ij})$, if $i=1, \ldots, n$.

Proof: See the Appendix.

Some helpful observations follow from this lemma. For instance, it appears from (i) of the lemma that the summation of the relative contributions of the coefficients d_{ij} over i and j amounts to zero. Thus, a same percentage change in all coefficients d_{ij} has no effect on r_{gh}. Obviously, the same applies to the coefficients a_{ij} and b_{ij}. Next, we observe from (ii) that instead of interpreting the sums $\sum_i w(r_{gh}, d_{ij})$, we can equivalently consider the sums $\sum_i w(r_{gh}, b_{ij})$ for different j. Further, it follows from (iv) that the relative contributions pertaining to the structural form coefficients of a model equation, which does not contain lagged endogenous variables, sum up to zero.

Concluding, we remark that the above lemma can be considered as an extension to the context of the eigenvectors, of a comparable lemma which holds with respect to the eigenvalues of the matrix D as given in SCHOONBEEK [1984]. Thus, it appears that the sensitivity of the eigenvectors and eigenvalues can be treated in an uniform way.

4 Application to a Linear Macro-Economic Model

In this section we will analyze the eigenvectors pertaining to a linear (Keynesian) economic model, which corresponds to the systematic part of the model I of KLEIN [1950]. In the first 3 equations the 2SLS estimated coefficient values are used. The estimation period is 1921-1941. (See for more details GOLDBERGER [1964]). The model reads

$$C_t = 0.017\ P_t + 0.216\ P_{t-1} + 0.810\ W_t + 16.555$$

$$I_t = 0.150\ P_t + 0.616\ P_{t-1} - 0.158\ K_{t-1} + 20.278$$

$$W_{1t} = 0.439\ E_t + 0.147\ E_{t-1} + 0.130\ A_t + 1.500$$

$$Y_t = C_t + I_t + G_t - T_t$$

$$P_t = Y_t - W_t$$

$$K_t = I_t + K_{t-1}$$

$$W_t = W_{1t} + W_{2t}$$
$$E_t = Y_t + T_t - W_{2t}.$$

The endogenous variables are: C_t (consumption), I_t (investment), W_{1t} (private wage bill), Y_t (national income), P_t (profits), K_t (end-of-year-capital stock), W_t (total wage bill), E_t (private product). The exogenous variables are: 1 (constant term), A_t (time trend $t-1931$, representing the influence of the labor unions), G_t (government expenditures), T_t (indirect taxes), W_{2t} (government wage bill). All variables (except 1 and A) are measured in 1934 US dollars. Formally the model can be written in the form of formula (1), where $y_t = (C, I, W_1, P, Y, K, W, E)'$ and $x_t = (1, W_2, T, G, A)'$. There are three lagged endogenous variables, i. e. P_{-1}, K_{-1} and E_{-1}.

The eigenvalues of the matrix D associated to the model are: $\lambda_1, \lambda_2 = 0.760 \pm 0.349\, i$, $\lambda_3 = 0.298$, $\lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 = 0$. The matrix D is diagonable. Now, we follow the argument of section 2 and investigate whether good approximations of the matrices H(1) and F can be obtained using the spectral decomposition of the matrix D. In table 1, we give the matrix F pertaining to the model, as computed by us. We have calculated $\|H(1)\|_2 = 126.398$ and $\|F\|_2 = 218.134$. In table 2, we list the computed values of the spectral norms associated with $(H(1) - H^G(1))$ and $(F - F^G)$ for all relevant sets of indices G. It appears that the quality of the approximations of the matrices H(1) and F depends crucially on the inclusion of the elements 1 and 2 in the set G. A good result is obtained when G=(1, 2, 3). Thus, we conclude that the eigenvectors r_1, r_2 and r_3 are most influential. The vector r_1 is computed as (0.29-0.04 i, 0.20+0.04 i, 0.29-0.04 i, 0.49, 0.20+0.03 i, 0.02−0.42 i, 0.29−0.04 i, 0.49)′. The right eigenvector r_2 is equal to $\bar{r}_1$, whereas r_3 is (0.38, 0.09, 0.44, 0.47, 0.03, −0.04, 0.44, 0.47)′. The sums of the relative contributions, discussed in section 3, pertaining to these vectors and the three lagged endogenous variables are presented in table 3. The relative contributions pertaining to r_2 are the complex conjugates of those of r_1. First consider r_1. It appears that the element r_{16}, corresponding to the variable K, is the most sensitive to changes in the model coefficients. The coefficients pertaining to P_{-1} and K_{-1} have a relatively large impact (with an opposite sign) on the real part of this element, whereas the coefficients of P_{-1} and E_{-1} have a relatively large impact (with an opposite sign) on the imaginary part of this element. We also note that the coefficients pertaining to E_{-1} have no impact on the imaginary parts of r_{14} and r_{18}, corresponding to the variables Y and E respectively. In general, we see that the effects on the imaginary parts of r_1 caused by changes in the coefficients related to P_{-1} and K_{-1} respectively, have an opposite sign. Now consider the eigenvector r_3. From table 3, we observe that the coefficients pertaining to the variable K_{-1} only have an impact on the elements r_{35} and r_{36}, corresponding to the variables P and K respectively. The effects of changes in the coefficients related to P_{-1} and E_{-1} always have an opposite sign here. In general, the elements of this vector are rather insensitive to the changes in the model coefficients.

TABLE 1

The Matrix* F *and the Corresponding Variables

	Const.	W_2	G	T	A
C.	40.62	0.56	1.33	−0.54	0.17
I.	0.00	0.00	0.00	0.00	0.00
W_1.	25.30	−0.26	1.37	−0.32	0.23
Y.	40.62	0.56	2.33	−1.54	0.17
P.	15.32	−0.18	0.96	−1.22	−0.06
K.	202.60	−0.89	4.68	−5.94	−0.28
W.	25.30	0.74	1.37	−0.32	0.23
E.	40.62	−0.44	2.33	−0.54	0.17

TABLE 2

The Values of $\|H(1)-H^G(1)\|_2$ ***and*** $\|F-F^G\|_2$ ***for Different Sets*** G

Elements of G	$\|H(1)-H^G(1)\|_2$	$\|F-F^G\|_2$
1, 2.	66.647	97.410
3.	189.241	265.806
4, 5, 6, 7, 8.	126.223	217.017
1, 2, 3.	10.386	10.295
1, 2, 4, 5, 6, 7, 8.	68.120	99.100
3, 4, 5, 6, 7, 8.	188.308	264.075

TABLE 3

The Sums $\sum_i w(r_{gh}, d_{ij})$ ***Associated with*** P_{-1}, K_{-1} ***and*** E_{-1} ($j=5, 6, 7$ ***respectively) and*** $g=1, 3$

	real parts $g=1$			imag. parts $g=1$			$g=3$		
h	P_{-1}	K_{-1}	E_{-1}	P_{-1}	K_{-1}	E_{-1}	P_{-1}	K_{-1}	E_{-1}
1.	−0.05	−0.02	0.06	−0.06	0.08	−0.02	0.01	0.00	−0.02
2.	−0.01	0.05	−0.04	−0.05	0.03	0.02	−0.04	0.00	0.05
3.	−0.09	−0.01	0.09	−0.04	0.08	−0.05	0.03	0.00	−0.02
4.	−0.06	0.04	0.02	−0.10	0.10	0.00	−0.02	0.00	0.02
5.	0.03	0.04	−0.08	−0.07	0.02	0.04	−0.05	0.01	0.04
6.	0.43	−0.44	0.01	−0.26	0.08	0.18	0.03	0.05	−0.07
7.	−0.09	−0.01	0.09	−0.04	0.08	−0.05	0.03	0.00	−0.02
8.	−0.06	0.04	0.02	−0.10	0.10	0.00	−0.02	0.00	0.02

5 Summary and Conclusion

We have analyzed in this paper the eigenvectors of dynamic linear economic models. In section 2, we discussed a procedure to identify the eigenvectors which contribute relatively much to the values of the endogenous variables of a given model. Although our discussion is given in terms of a first-order model, the analysis readily generalizes to higher-order models. The sensitivity of the eigenvectors with respect to changes in the model coefficients is investigated in section 3. Our approach here extends in an uniform way the sensitivity analysis of the eigenvalues of our models, as known from the literature. For illustrative purposes, we discussed in section 4 an application of our procedures to a small economic model.

APPENDIX

In this appendix we give the proof of the lemma of section 3. For notational ease, set $g=1$. From MAGNUS [1986, formula (18)] we obtain the following expression for the first derivative dr_1 of the eigenvector r_1 with respect to (small) changes $d\mathrm{D}$ in the matrix D

$$dr_1 = (\lambda_1 \mathrm{I} - \mathrm{D})^+ (\mathrm{I} - r_1 l_1')(d\mathrm{D})\, r_1$$

where $(\lambda_1 \mathrm{I} - \mathrm{D})^+$ is the Moore-Penrose inverse of the matrix $(\lambda_1 \mathrm{I} - \mathrm{D})$. Let us define $(\lambda_1 \mathrm{I} - \mathrm{D})^+ \equiv \mathrm{X} = (x_{rs})$ and let I_{ij} represent the $(n \times n)$ matrix which has unity in the (i, j)-th position and zeroes elsewhere.

– (i)

If we set $d\mathrm{D} = \mathrm{I}_{ij}$ then

$$\sum_i \sum_j w(r_{1\,h}, d_{ij}) = \sum_i \sum_j [x_{hi} r_{1\,j} - \sum_s x_{hs} r_{1\,s} l_{1\,i} r_{1\,j}]\, d_{ij}$$

$$= \sum_i [x_{hi} \lambda_1 r_{1\,i} - \sum_s x_{hs} r_{1\,s} l_{1\,i} \lambda_1 r_{1\,i}]$$

$$= \sum_i \lambda_1 x_{hi} r_{1\,i} - \sum_s \lambda_1 x_{hs} r_{1\,s} = 0.$$

– (ii)

$$\sum_i w(r_{1\,h}, d_{ij}) = \sum_i [x_{hi} r_{1\,j} - \sum_s x_{hs} r_{1\,s} l_{1\,i} r_{1\,j}]\, d_{ij}$$

$$= \sum_i x_{hi} r_{1\,j} d_{ij} - \sum_s x_{hs} r_{1\,s} r_{1\,j} \lambda_1 l_{1\,j}.$$

Define $A^{-1} \equiv \tilde{A} = (\tilde{a}_{ij})$. If $dB = I_{ij}$ then $dD = \tilde{A}\, I_{ij}$.

It follows that

$$\sum_i w(r_{1\,h}, b_{ij}) = \sum_i [\sum_s x_{hs} \tilde{a}_{si} r_{1\,j} - \sum_t x_{ht} r_{1\,t} \sum_q l_{1\,q} \tilde{a}_{qi} r_{1\,j}]\, b_{ij}$$

$$= \sum_s x_{hs} d_{sj} r_{1j} - \sum_t x_{ht} r_{1\,t} \sum_q l_{1\,q} d_{qj} r_{1\,j}$$

$$= \sum_s x_{hs} d_{sj} r_{1\,j} - \sum_t x_{ht} r_{1\,t} \lambda_1 l_{1\,j} r_{1\,j}$$

which proves (ii).

(iii)

$$\sum_j w(r_{1\,h}, d_{ij}) = \sum_j [x_{hi} r_{1\,j} - \sum_s x_{hs} r_{1\,s} l_{1\,i} r_{1\,j}]\, d_{ij}$$

$$= \lambda_1 x_{hi} r_{1\,i} - \sum_s x_{hs} r_{1\,s} l_{1\,i} \lambda_1 r_{1\,i}.$$

Now observe that if $dA = I_{ij}$, then $dD = -\tilde{A}\, I_{ij} \tilde{A} B = -\tilde{A}\, I_{ij} D$.

Thus

$$\sum_i w(r_{1\,h}, a_{ij}) \quad = -\sum_i [\sum_s x_{hs} \tilde{a}_{si} \lambda_1 r_{1\,j} - \sum_t x_{ht} r_{1\,t} \sum_q l_{1\,q} \tilde{a}_{qi} \lambda_1 r_{1\,j}]\, a_{ij}$$

$$= -[x_{hj} \lambda_1 r_{1\,j} - \sum_t x_{ht} r_{1\,t} l_{1\,j} \lambda_1 r_{1\,j}].$$

Statement (iii) follows immediately.

–(iv)

$$\sum_j w(r_{1\,h}, b_{ij}) \quad = \sum_j [\sum_s x_{hs} \tilde{a}_{si} r_{1\,j} - \sum_t x_{ht} r_{1\,t} \sum_q l_{1\,q} \tilde{a}_{qi} r_{1\,j}]\, b_{ij}$$

$$= \sum_j [\sum_s x_{hs} \tilde{a}_{si} \lambda_1 a_{ij} r_{1\,j} - \sum_t x_{ht} r_{1\,t} \sum_q l_{1\,q} \tilde{a}_{qi} \lambda_1 a_{ij} r_{1\,j}]$$

$$= -\sum_j w(r_{1\,h}, a_{ij}).$$

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